Advanced Structural Geology, Fall 2022

3D stress

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Mostly from Ragan and Schultz, unpublished

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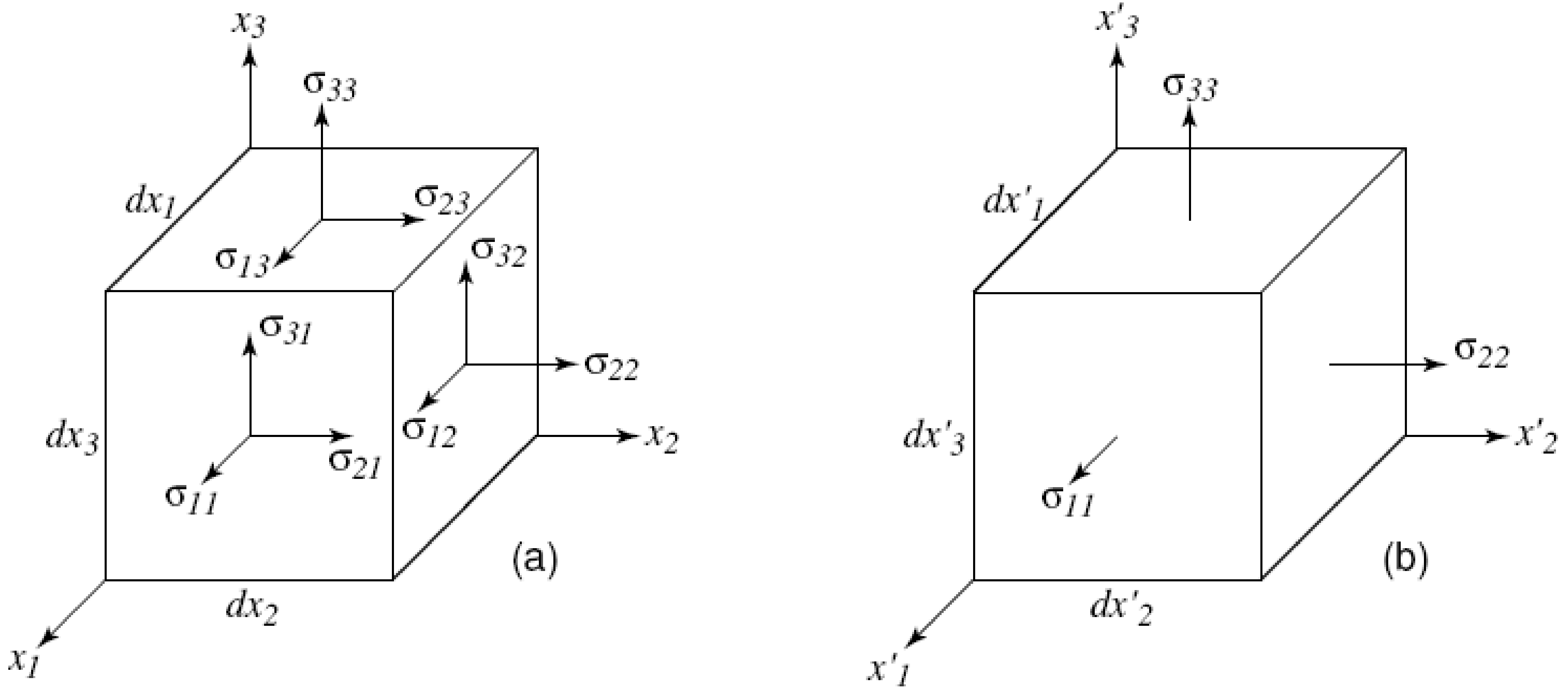


Figure 13.2: Cartesian stress components: (a) general components;(b) principal stresses.

Note sign convention change

In a special set of transformed axes, all the shearing components vanish (Fig. 13.2b). The matrix representation is then

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

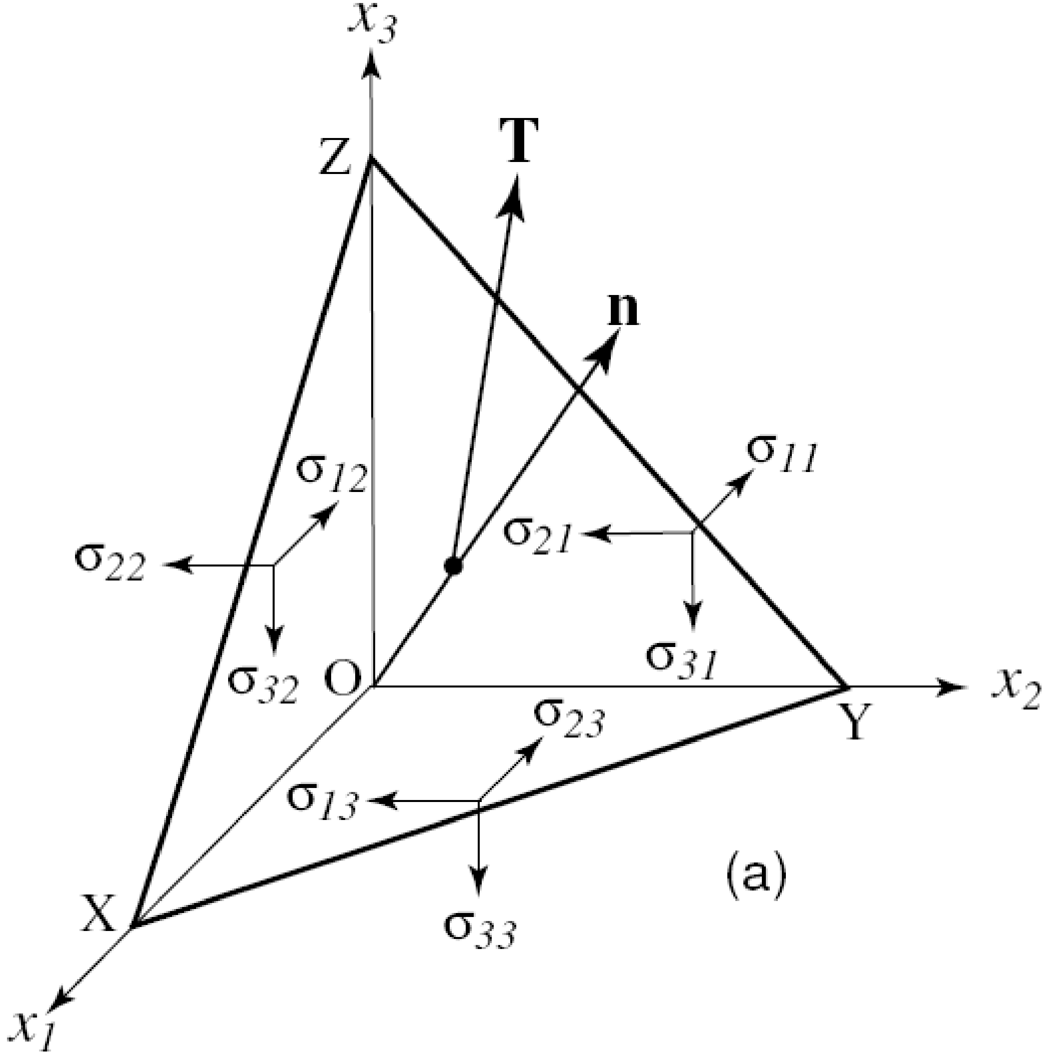
The three remaining components are the *principal stresses* $\sigma_1 \geq \sigma_2 \geq \sigma_3$. These are the eigenvalues and their orientations are the corresponding eigenvectors of the stress tensor. As we see later, the use of this diagonal matrix wherever possible greatly simplifies many problems.

With this tensor sign convention

$$\sigma_1 = \text{the greatest tensile stress or least compressive stress,}$$
$$\sigma_3 = \text{the least tensile stress or greatest compressive stress.}$$

Note sign convention change

Tractions from stress



(a)

Tractions from stress

$$T_i^{(n)} = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3$$

$$T_i^{(n)} = \sigma_{ij} n_j$$

$$T_1^{(n)} = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

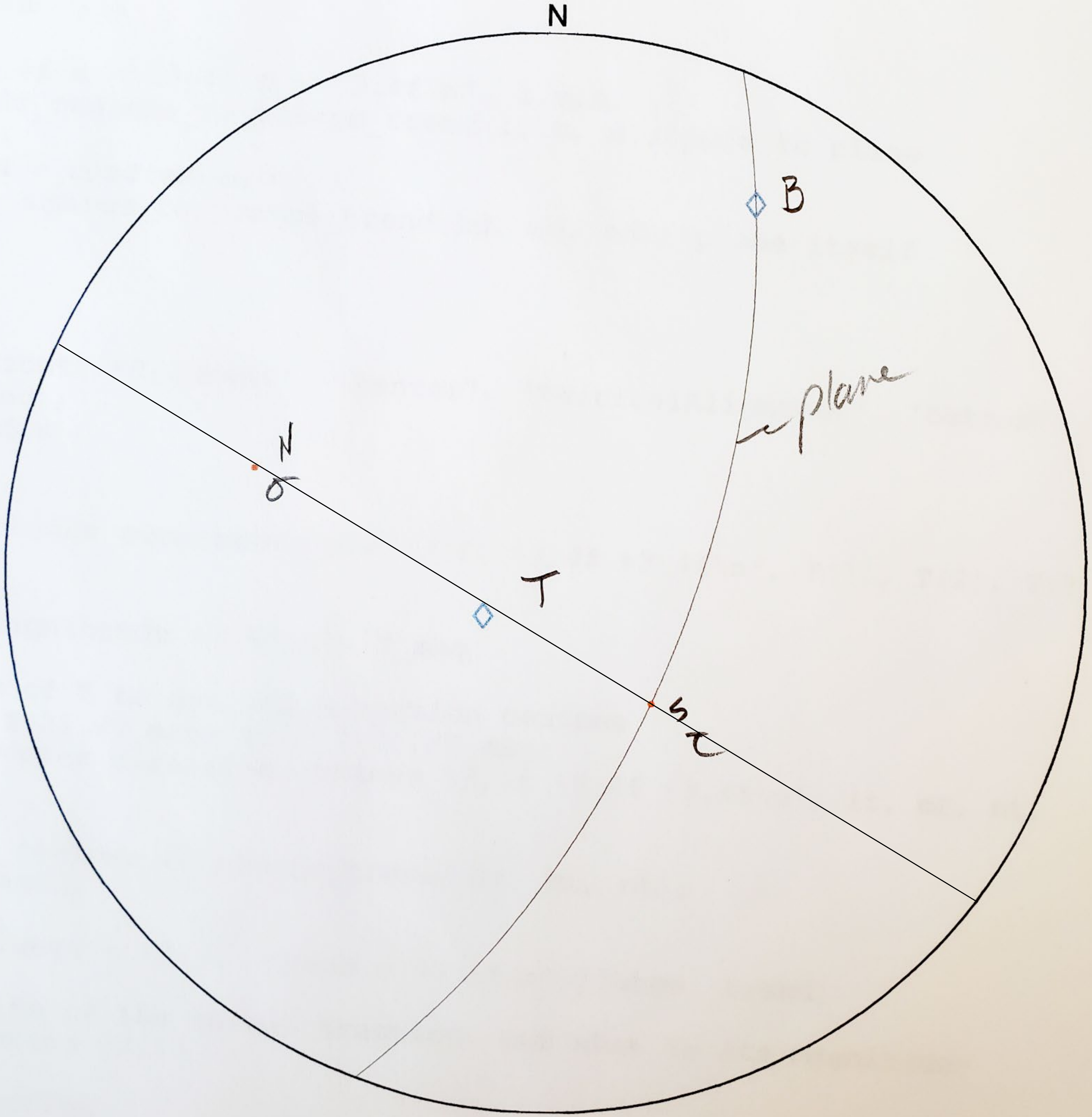
$$T_2^{(n)} = \sigma_{21} n_1 \quad \sigma_{22} n_2 \quad \sigma_{23} n_3$$

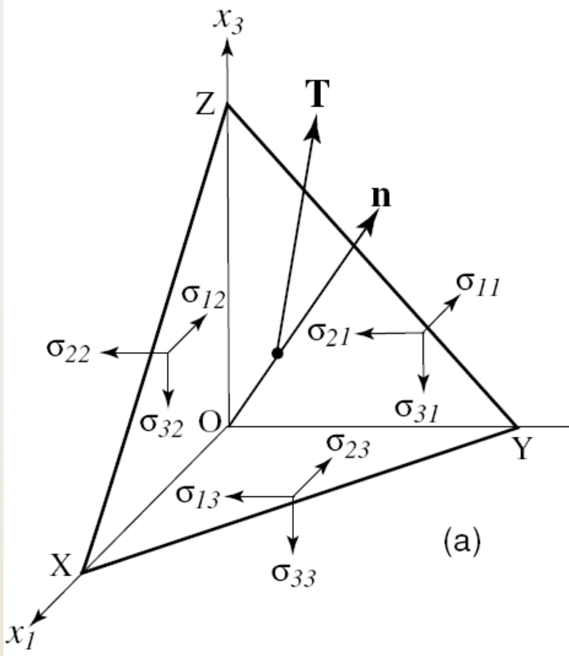
$$T_3^{(n)} = \sigma_{31} n_1 + \sigma_{32} n_2 \quad \sigma_{33} n_3$$

$$\vec{T}(\vec{n}) = \sigma \vec{n}$$

$$\begin{bmatrix} T_1^{(n)} \\ T_2^{(n)} \\ T_3^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} \cdot n_3 \\ \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{bmatrix}$$

T=S*N (MATLAB matrix multiplication)





(a)

Traction normal and shear components

$$\sigma \equiv T_N^{(\mathbf{n})} \quad \text{and} \quad \tau \equiv T_S^{(\mathbf{n})}.$$

The magnitude of the normal component is obtained from the projection of the traction vector onto the normal vector by

$$\sigma \equiv T_N^{(\mathbf{n})} \quad \text{and} \quad \tau \equiv T_S^{(\mathbf{n})}.$$

or compactly

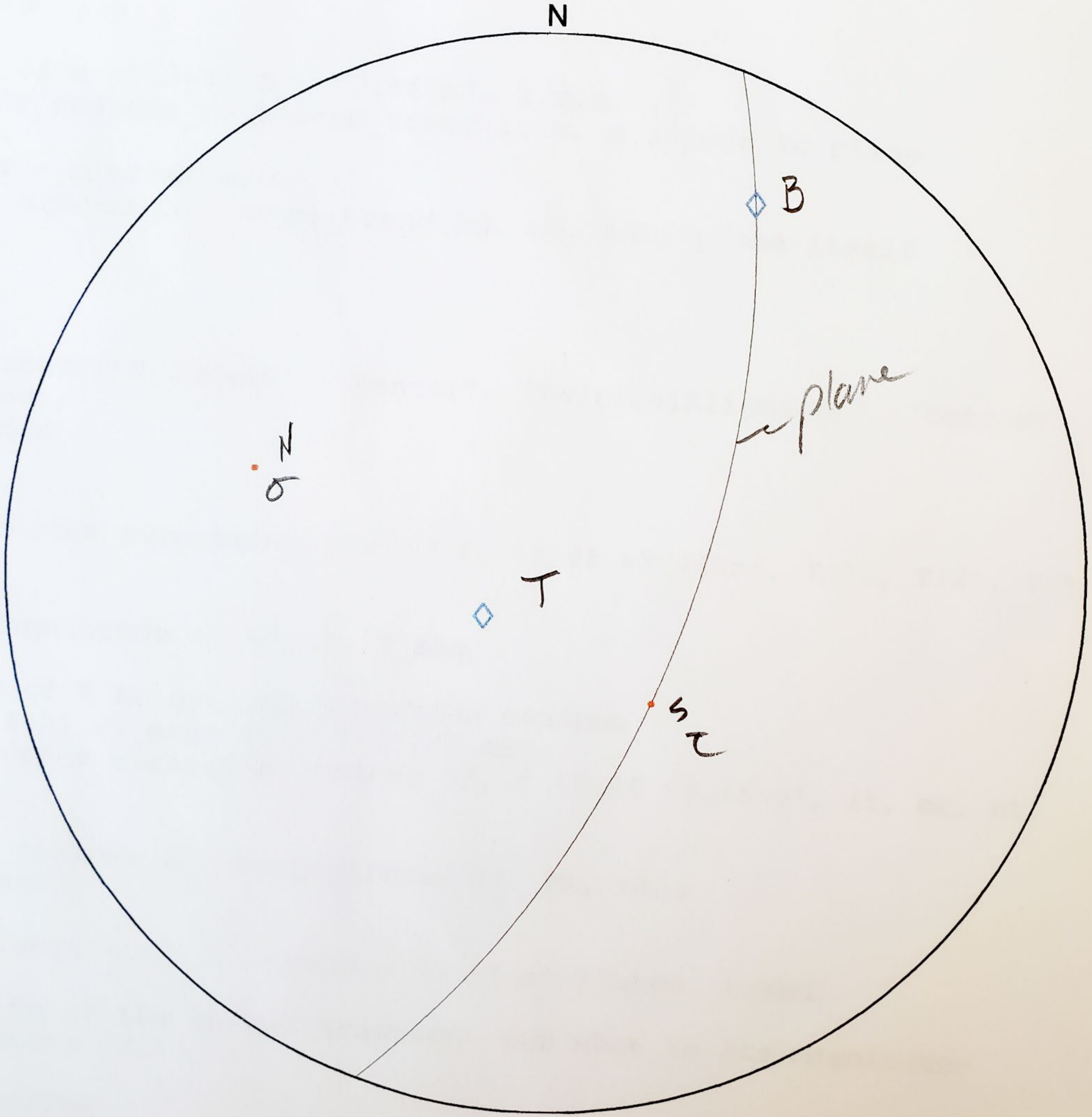
$$\sigma = T_i n_i.$$

sigma=dot(T,N) (MATLAB)

There is a simple formula which uses vector products which encapsulates this entire graphical construction (McKenzie, 1969, p. 594).

$$\mathbf{T}_S = \mathbf{n} \times [\mathbf{T}^{(\mathbf{n})} \times \mathbf{n}]$$

The product $[\mathbf{T} \times \mathbf{n}]$ is a vector normal to the plane of $\mathbf{T}^{(\mathbf{n})}$ and $\mathbf{n}$. It is also the direction of $\tau = 0$ in the plane, called the *null direction*, and commonly given the symbol $\mathbf{B}$. The product $\mathbf{n} \times \mathbf{B}$ then gives the vector in the direction of the line of intersection of the plane of $\mathbf{Tn}$ and the plane of interest and this defines the direction and magnitude of the maximum shear.


N
B
plane
σN
T
στ

Problem

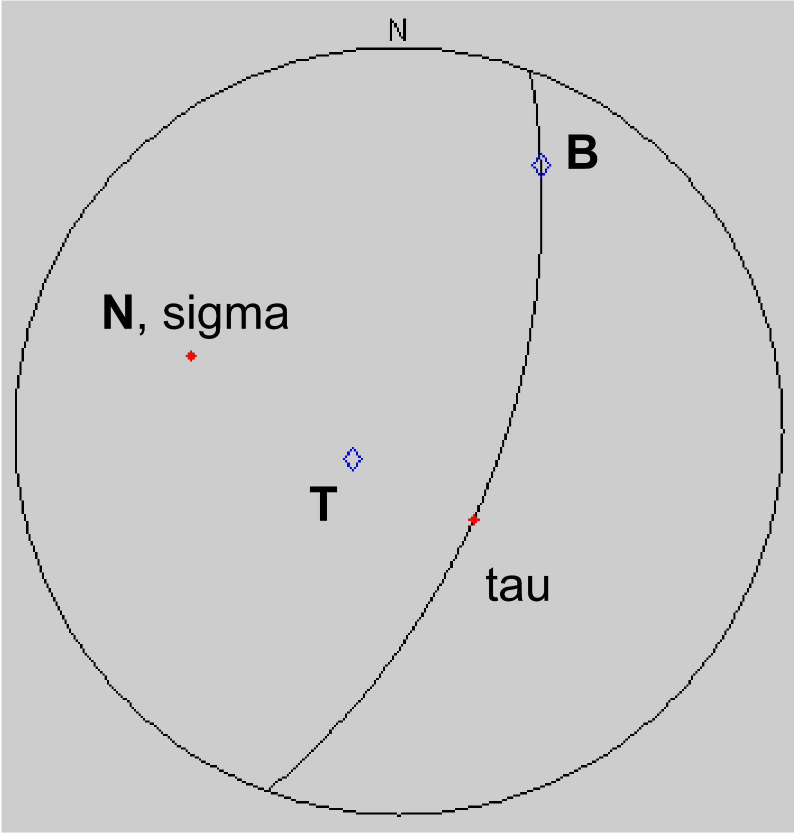
- What are the two components of the traction acting on a plane whose pole has a plunge and trend of attitude of 30/290, given the following stress components?

$$\boldsymbol{\sigma} = \begin{bmatrix} -50 & -20 & 10 \\ -20 & -30 & -15 \\ 10 & -15 & -120 \end{bmatrix} \text{ MPa.}$$

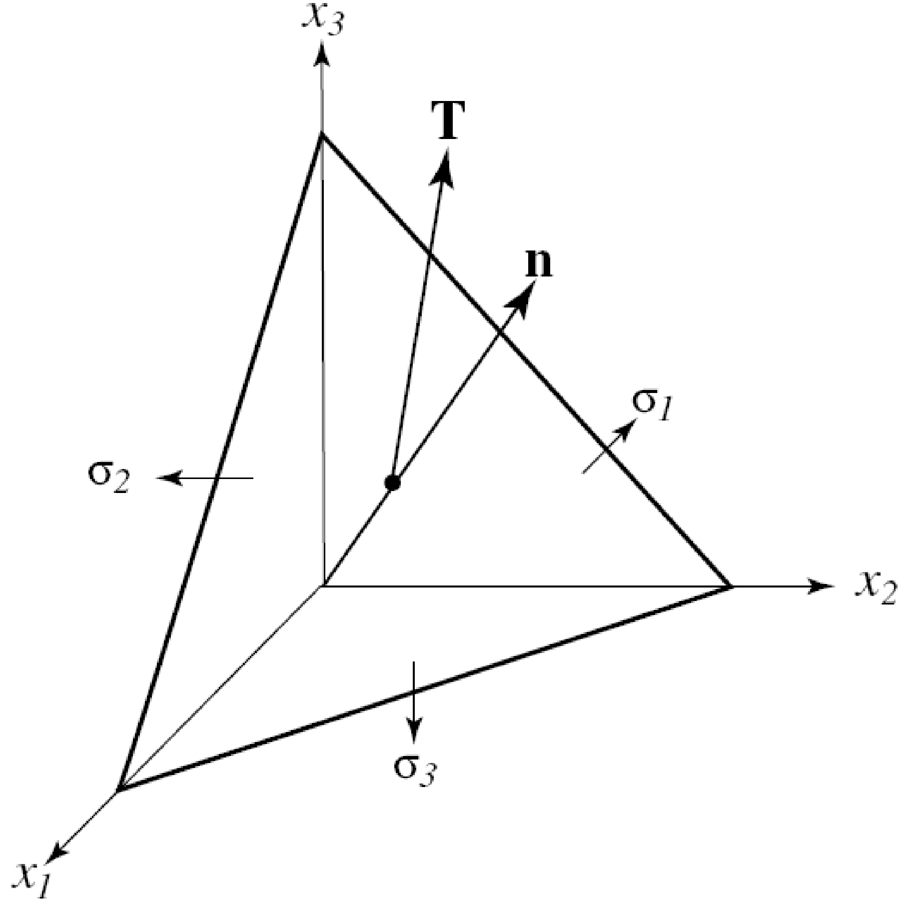
```
S = [-50 -20 10; -20 -30 -15; 10 -15 -120]

[l,m,n] =plunge_trend_to_dir_cosines(30,290);

N=[l;m;n];

T=S*N; %equation 13.11
T_mag = sqrt(sum(T.^2));

%normalize components of T to get its direction
cosines
lt=T(1)./T_mag; mt = T(2)./T_mag; nt =
T(3)./T_mag;

%plot traction vector
[plunge, trend] =
dir_cosines_to_plunge_trend2(lt, mt, nt);
```

```
%we know the orientation of the normal
traction, but what is its magnitude?
sigma = dot(T,N);%equation 13.13

%Now for the shear traction; use the
McKenzie construction
B = cross(T,N); %vector normal to the
plane containing T and N
B_mag = sqrt(B(1)^2 + B(2)^2 + B(3)^2);
lb = B(1)./B_mag;
mb = B(2)./B_mag;
nb = B(3)./B_mag;

Ts = cross(N,B); %shear traction direction
Ts_mag = sqrt(Ts(1)^2 + Ts(2)^2 +
Ts(3)^2);
Ts(1) = Ts(1)./Ts_mag;
Ts(2) = Ts(2)./Ts_mag;
Ts(3) = Ts(3)./Ts_mag;

%let's check that the normal and shear are
components of the traction
testmag = sqrt(sum(sigma.^2 + Ts_mag.^2));
```

traction vector components are 6.4660 10.9900 -44.8311
traction magnitude 46.6091
traction vector direction cosines 0.1387 0.2358 -0.9619
traction plunge = 74.1 trend = 239.5
normal traction mag -29.44 shear traction mag 36.13
check that components make same length as traction:
46.6091 =?= 46.6091



Traction from principal stresses



Traction from principal stresses
some simplification from the prior analysis

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \sigma \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{bmatrix}$$

n_i are principal axes

$$T^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

Normal traction is $\sigma = N \cdot T$

Problem

- With the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 120 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 20 \end{bmatrix} \text{MPa}$$

determine the normal and shearing components of the traction vector acting on a plane whose normal is given by the direction angles $\alpha_1 = 45°$ and $\alpha_2 = 60°$.

```
alpha1 = 45; alpha2 = 60;

S = [120 0 0; 0 80 0; 0 0 20]
l = cosd(alpha1);
m = cosd(alpha2);
n = sqrt(1 - l.^2 - m.^2);
N =[l;m;n];
[plunge, trend] = dir_cosines_to_plunge_trend(l, m,
n);%pole to plane

ld = -l; md = -m; nd = cosd(asind(n));
[dip, dipdir] = dir_cosines_to_plunge_trend(ld, md,
nd);%plane itself


T=S*N; %equation 13.11

T_mag = sqrt(sum(T.^2));

%normalize components of T to get its direction
cosines
lt=T(1)./T_mag; mt = T(2)./T_mag; nt = T(3)./T_mag;

%plot traction vector
[plunge, trend] = dir_cosines_to_plunge_trend2(lt,
mt, nt);
plotdiamond(plunge,trend);
```

```
%we know the orientation of the normal
traction, but what is its magnitude?
sigma = dot(T,N);%equation 13.13

%Now for the shear traction use; the
McKenzie construction
B = cross(T,N); %vector normal to the plane
containing T and N
B_mag = sqrt(B(1)^2 + B(2)^2 + B(3)^2);
lb = B(1)./B_mag;
mb = B(2)./B_mag;
nb = B(3)./B_mag;

Ts = cross(N,B); %shear traction direction
Ts_mag = sqrt(Ts(1)^2 + Ts(2)^2 + Ts(3)^2);
Ts(1) = Ts(1)./Ts_mag;
Ts(2) = Ts(2)./Ts_mag;
Ts(3) = Ts(3)./Ts_mag;
```

S =

120	0	0
0	80	0
0	0	20

P l = 0.7071 m = 0.5000 n = 0.5000

traction vector components are 84.8528 40.0000 10.0000

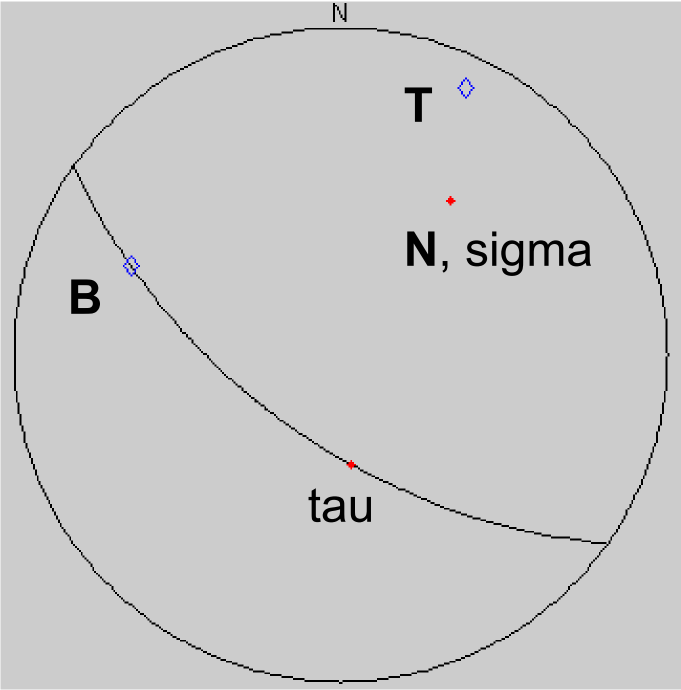
traction magnitude 94.3398

traction vector direction cosines 0.8994 0.4240 0.1060

traction plunge = 6.1 trend = 25.2

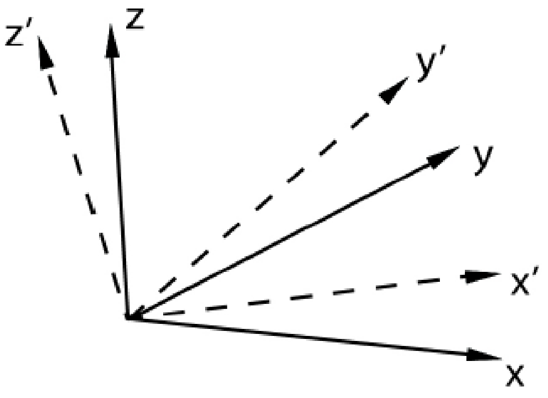
normal traction mag 85.00 shear traction mag 40.93

check that components make same length as traction: 94.3398 =?= 94.3398



Full stress tensor rotation

Rotation of stress tensor

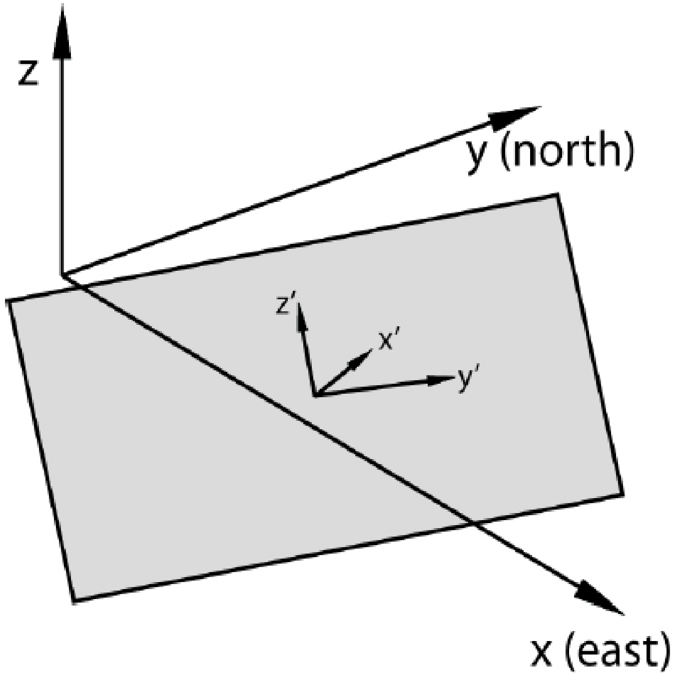


x'y'z' - new coordinate system

xyz - old coordinate system

stress tensor in old coordinate system

$$\sigma_{x,y,z} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}$$



Rotation from old in new coordinate system:

$$[\sigma]_{x',y',z'} = R^T [\sigma]_{x,y,z} R$$

Directional cosines (dot-product)

$l = x*x'$; $l' = x*y'$; $l'' = x*z'$

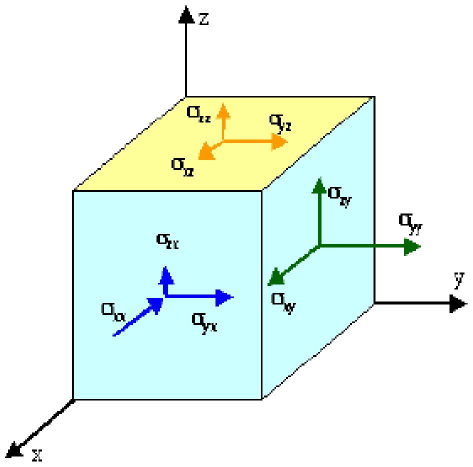
$m = y*x'$; $m' = y*y'$; $m'' = y*z'$

$n = z*x'$; $n' = z*y'$; $n'' = z*z'$

R -Rotation matrix

$$R = \begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix}$$

$$R^T = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}$$



-O. Zielke

Given $\mathbf{S}$ principal stress tensor with orientation $x'y'z'$

Rotate to N-S, E-W components

$$\mathbf{R} = \begin{bmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{bmatrix} \text{ where } \begin{array}{rl} l & = x * x' \\ l' & = x * y' \\ l'' & = x * z' \\ \ldots & \end{array}$$

$$\mathbf{S}' = \mathbf{R}^T \mathbf{S} \mathbf{R}$$

And given plane with normal vector direction cosines $\mathbf{N}$

Traction $\mathbf{T} = \mathbf{S}' * \mathbf{N}$ (row and column multiplication)
$T = \sqrt{\mathbf{T}(1)^2 + \mathbf{T}(2)^2 + \mathbf{T}(3)^2}$ traction magnitude

$\sigma_n = \mathbf{T} \cdot \mathbf{N}$ dot product for normal traction magnitude

$\mathbf{B} = \mathbf{T} \times \mathbf{N}$ cross product for null vector
$B = \sqrt{\mathbf{B}(1)^2 + \mathbf{B}(2)^2 + \mathbf{B}(3)^2}$ $\mathbf{B}$ magnitude
$\mathbf{B}_{normalized} = \mathbf{B}./B$ normalize for orientation if necessary

$\mathbf{T_s} = \mathbf{N} \times \mathbf{B}$ cross product for shear traction vector
$\tau = \sqrt{\mathbf{T_s}(1)^2 + \mathbf{T_s}(2)^2 + \mathbf{T_s}(3)^2}$ shear traction magnitude
$\mathbf{T_s}_{normalized} = \mathbf{T_s}./\tau$ normalize for shear traction orientation

Coulomb failure function: $\Delta\sigma_f = \Delta\tau - (\mu - P)\Delta\sigma_n$

Full stress tensor rotation example

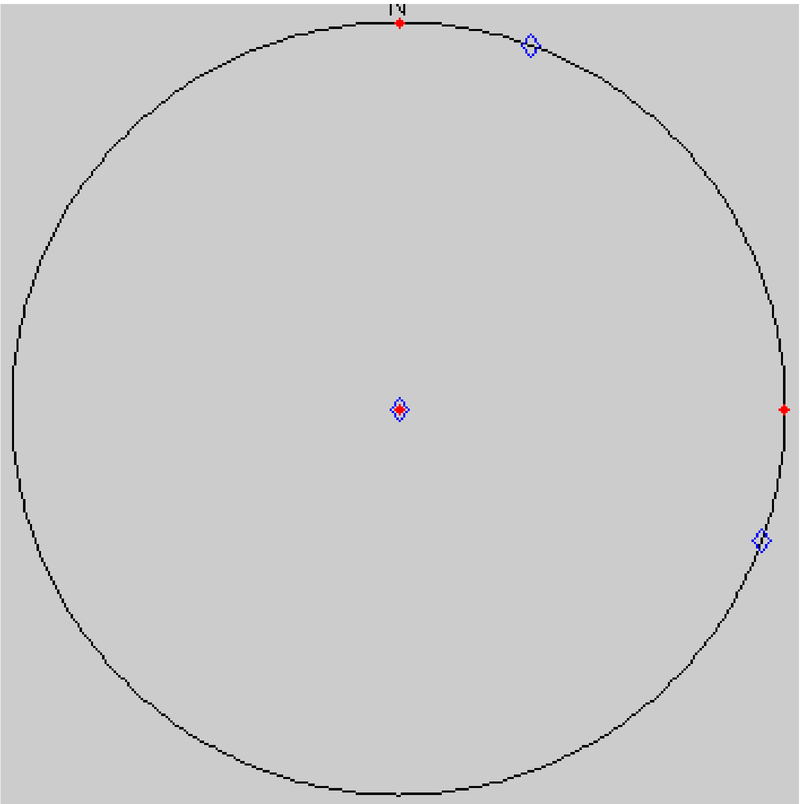
With σ_1 oriented 020, σ_2 oriented 110 and σ_3 vertical Given S =

S =

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{matrix}$$

-100	0	0
0	-50	0
0	0	-10

```
xprime l = 0.9397 m = 0.3420 n = 0.0000
yprime l = -0.3420 m = 0.9397 n = 0.0000
zprime l = -0.0000 m = -0.0000 n = 1.0000
checks for orthogonality:
       xy 0.0000 xz 0.0000 yz 0.0000

R =

    0.9397   -0.3420         0
    0.3420    0.9397         0
         0         0    1.0000


rotatedS =

  -94.1511   16.0697         0
   16.0697  -55.8489         0
         0         0  -10.0000
```



%**MATLAB main script**

```
%buildrotationmatrix2(xprimetrend, xprimeplunge, yprimetrend,yprimeplunge,zprimetrend,zprimeplunge, talkandplot)
 R = buildrotationmatrix2(         20,           0,         110,          0,         0,         90, 1)

S = [-100 0 0; 0 -50 0; 0 0 -10]

rotatedS = R'*S*R
```

function [R] = buildrotationmatrix2(xprimetrend, xprimeplunge, yprimetrend,yprimeplunge,zprimetrend,zprimeplunge, talkandplot)

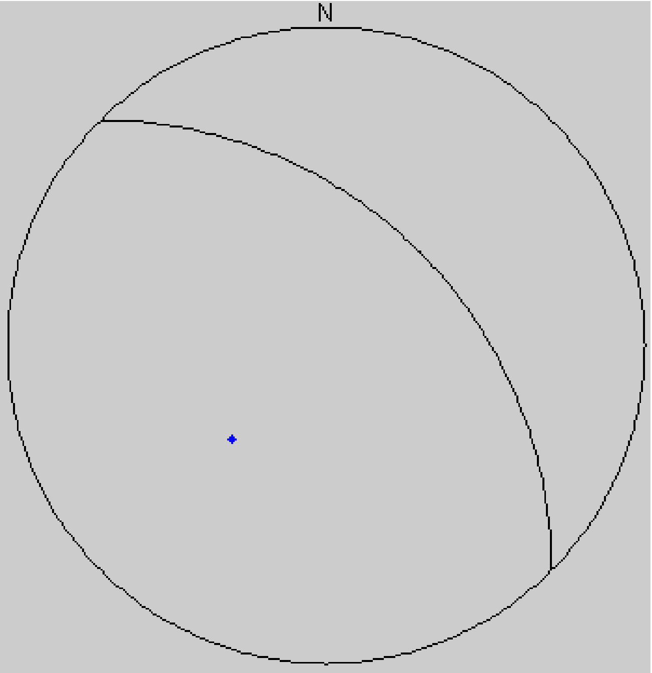
```
%This takes the orientation of the original coordinate system as trends and
%plunges and
%computes the direction cosines of the final coordinate system and then
%provides the rotation matrix of the form:
%     l l' l''
% R = m m' m''
%     n n' n"
%assumes the angles are in degrees
%assume that you want to rotate to north south
X = [1 0 0];
Y = [0 1 0];
Z = [0 0 1];

[xprime(1), xprime(2) xprime(3)] =plunge_trend_to_dir_cosines(xprimeplunge,xprimetrend);
[yprime(1), yprime(2) yprime(3)] =plunge_trend_to_dir_cosines(yprimeplunge,yprimetrend);
[zprime(1), zprime(2) zprime(3)] =plunge_trend_to_dir_cosines(zprimeplunge,zprimetrend);
%check to make sure you put them in right and they are orthogonal:
checkxy = dot(xprime,yprime);
checkxz = dot(xprime,zprime);
checkyz = dot(yprime,zprime);

if talkandplot==1
fprintf(1,'xprime l = %3.4f m = %3.4f n = %3.4f\n', xprime(1), xprime(2), xprime(3))
fprintf(1,'yprime l = %3.4f m = %3.4f n = %3.4f\n', yprime(1), yprime(2), yprime(3))
fprintf(1,'zprime l = %3.4f m = %3.4f n = %3.4f\n', zprime(1), zprime(2), zprime(3))
fprintf(1,'checks for orthogonality: xy %3.4f xz %3.4f yz %3.4f\n', checkxy, checkxz, checkyz)
```

```
primitive1;
text(0,1, 'N', 'HorizontalAlignment', 'center', 'VerticalAlignment', 'bottom')
plotdiamond(xprimeplunge,xprimetrend);
plotdiamond(yprimeplunge,yprimetrend);
plotdiamond(zprimeplunge,zprimetrend);
    %plot final coordinate system
    [plunge, trend] = dir_cosines_to_plunge_trend2(X(1), X(2), X(3));
    plotpoint(plunge,trend);
    [plunge, trend] = dir_cosines_to_plunge_trend2(Y(1), Y(2), Y(3));
    plotpoint(plunge,trend);
    [plunge, trend] = dir_cosines_to_plunge_trend2(Z(1), Z(2), Z(3));
    plotpoint(plunge,trend);
end
```

```
R(1,1) = X*xprime';
R(1,2) = X*yprime';
R(1,3) = X*zprime';
R(2,1) = Y*xprime';
R(2,2) = Y*yprime';
R(2,3) = Y*zprime';
R(3,1) = Z*xprime';
R(3,2) = Z*yprime';
R(3,3) = Z*zprime';
```

Example application to South Mountains faults

```
%Set up Receiver Fault
%We are interested in a plane 315,45 (RHR)
poleplunge=45;
poletrend =225;

[l,m,n]
=plunge_trend_to_dir_cosines(poleplunge,poletrend);
ld1 = -l; md1 = -m; nd1 = cosd(poleplunge);
[dip, dipdir] = dir_cosines_to_plunge_trend(ld1, md1,
nd1);
```



dip is 45.0 and dip dir is 45.0

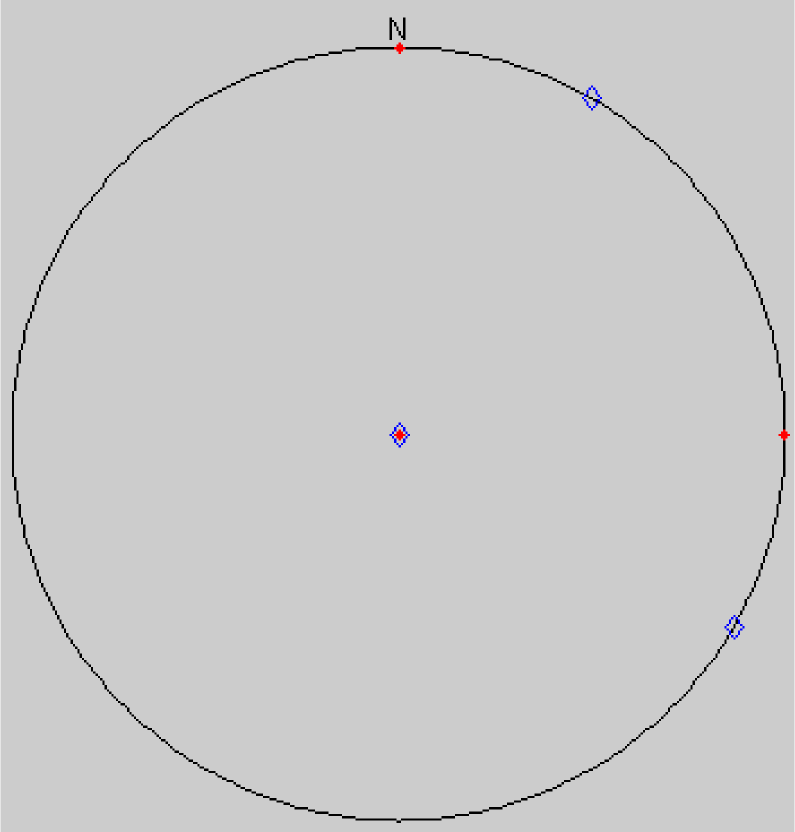
```
N=[l;m;n];
```

Example application to South Mountains faults

```
%Set up stress tensor
%assume that the principal stresses are appropriate for normal faulting
%conditions so maximum stress is the vertical stress
sv = -26.7.*12; %assume 26.5 MPa per km and 12 km depth
shmin = sv.*0.1; %assume the 1 direction is the minimum horizontal stress and is 10%
shmax = sv.*0.25; %assume the 2 direction is intermediate
S = [shmin 0 0;
         0 shmax 0;
         0   0 sv]
%buildrotationmatrix2(xprimetrend, xprimeplunge, yprimetrend,yprimeplunge,zprimetrend,zprimeplunge, talkandplot)
 R = buildrotationmatrix2(      30,             0,          120,             0,           0,           90, 1)
```

rotatedS = R'*S*R

```
S =

  -32.0400         0         0
         0  -80.1000         0
         0         0 -320.4000

xprime l = 0.8660 m = 0.5000 n = 0.0000
yprime l = -0.5000 m = 0.8660 n = 0.0000
zprime l = -0.0000 m = -0.0000 n = 1.0000
checks for orthogonality: xy 0.0000 xz 0.0000 yz 0.0000

R =

    0.8660   -0.5000         0
    0.5000    0.8660         0
         0         0    1.0000


rotatedS =

  -44.0550  -20.8106         0
  -20.8106  -68.0850         0
         0         0 -320.4000
```



Example application to South Mountains faults

```
%Now resolve the stresses
T=rotatedS*N; %equation 13.11

T_mag = sqrt(sum(T.^2));

%normalize components of T to get its direction cosines
lt=T(1)./T_mag; mt = T(2)./T_mag; nt = T(3)./T_mag;

%plot traction vector
[plunge, trend] = dir_cosines_to_plunge_trend2(lt, mt, nt);

%we know the orientation of the normal traction,
%but what is its magnitude?
sigma = dot(T,N); %equation 13.13
```

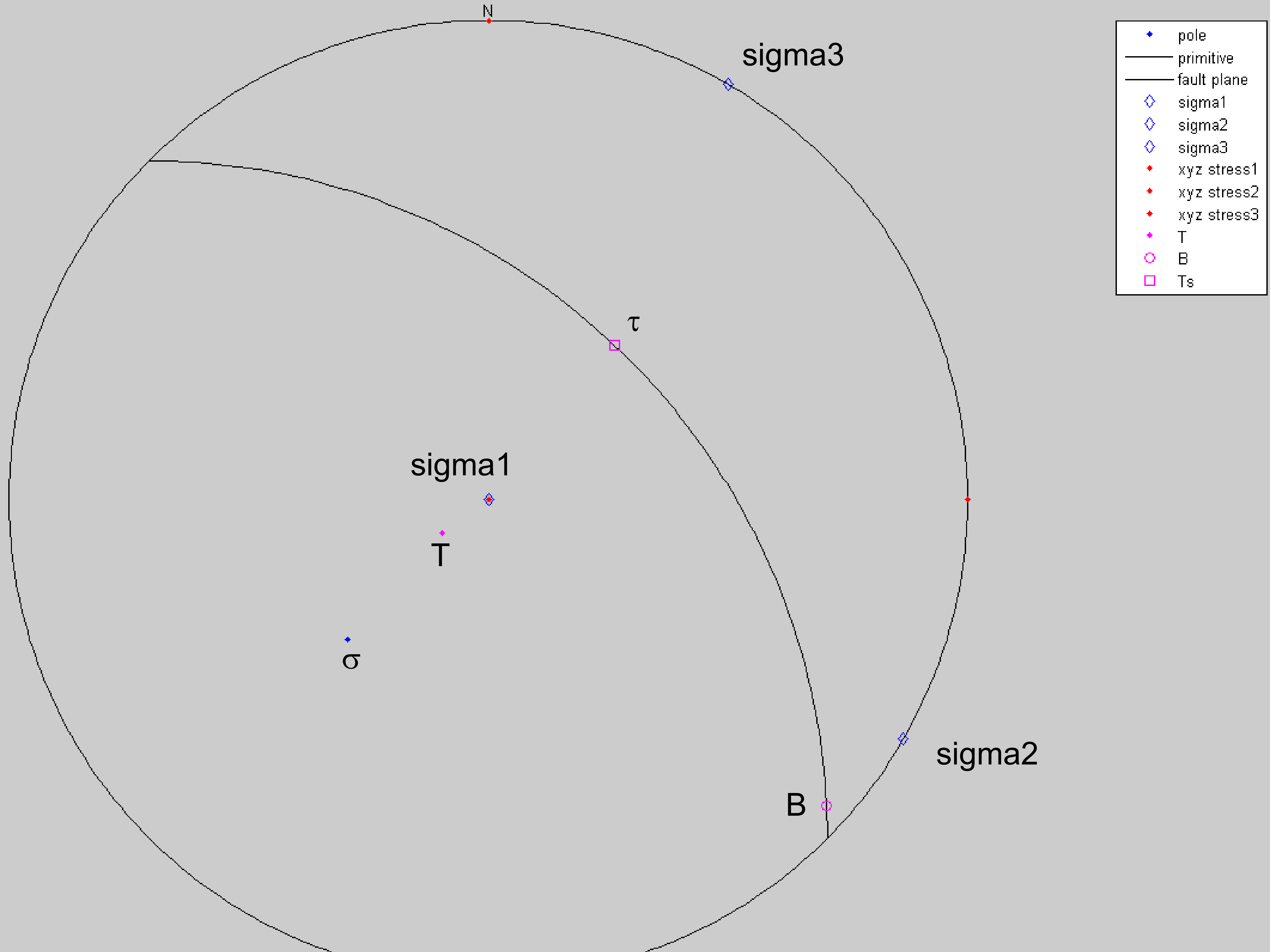
traction vector components are 32.4328 44.4478 -226.5570
traction magnitude 233.1428
traction vector direction cosines 0.1391 0.1906 -0.9718
traction plunge = 76.3 trend = 233.9
normal traction mag -198.64

Example application to South Mountains faults

```
%Now for the shear traction; use the McKenzie construction
B = cross(T,N); %vector normal to the plane containing T and N
B_mag = sqrt(B(1)^2 + B(2)^2 + B(3)^2);
lb = B(1)./B_mag;
mb = B(2)./B_mag;
nb = B(3)./B_mag;

[plunge, trend] = dir_cosines_to_plunge_trend2(lb,mb,nb);

Ts = cross(N,B); %shear traction direction
Ts_mag = sqrt(Ts(1)^2 + Ts(2)^2 + Ts(3)^2);
Ts(1) = Ts(1)./Ts_mag;
Ts(2) = Ts(2)./Ts_mag;
Ts(3) = Ts(3)./Ts_mag;
```

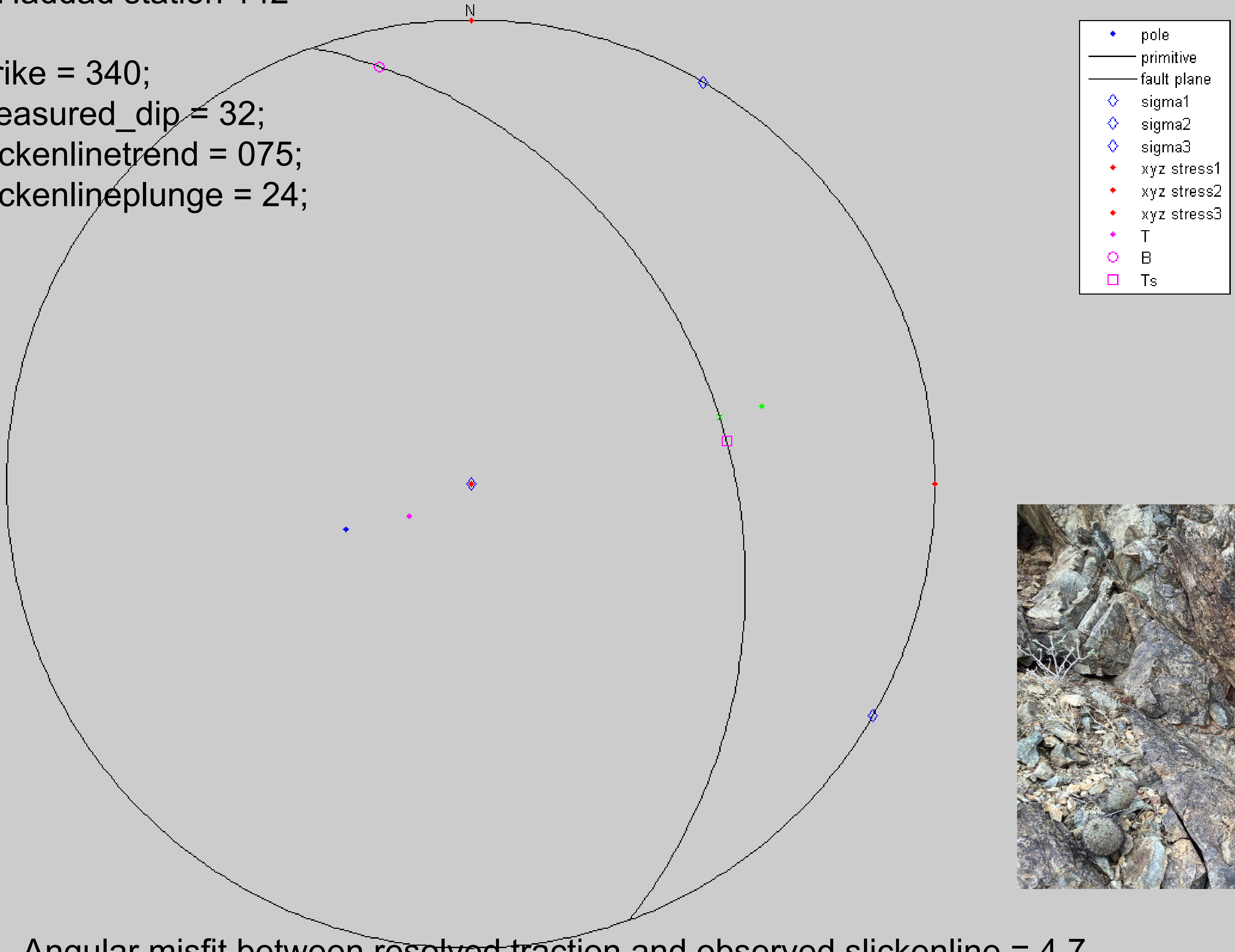
shear traction mag 122.06
check that components make
same length as traction: 233.1428 =?= 233.1428

```
[plunge, trend] = dir_cosines_to_plunge_trend2(Ts(1), Ts(2),
Ts(3));

%let's check that the normal and shear are components of the
traction
testmag = sqrt(sum(sigma.^2 + Ts_mag.^2));
```

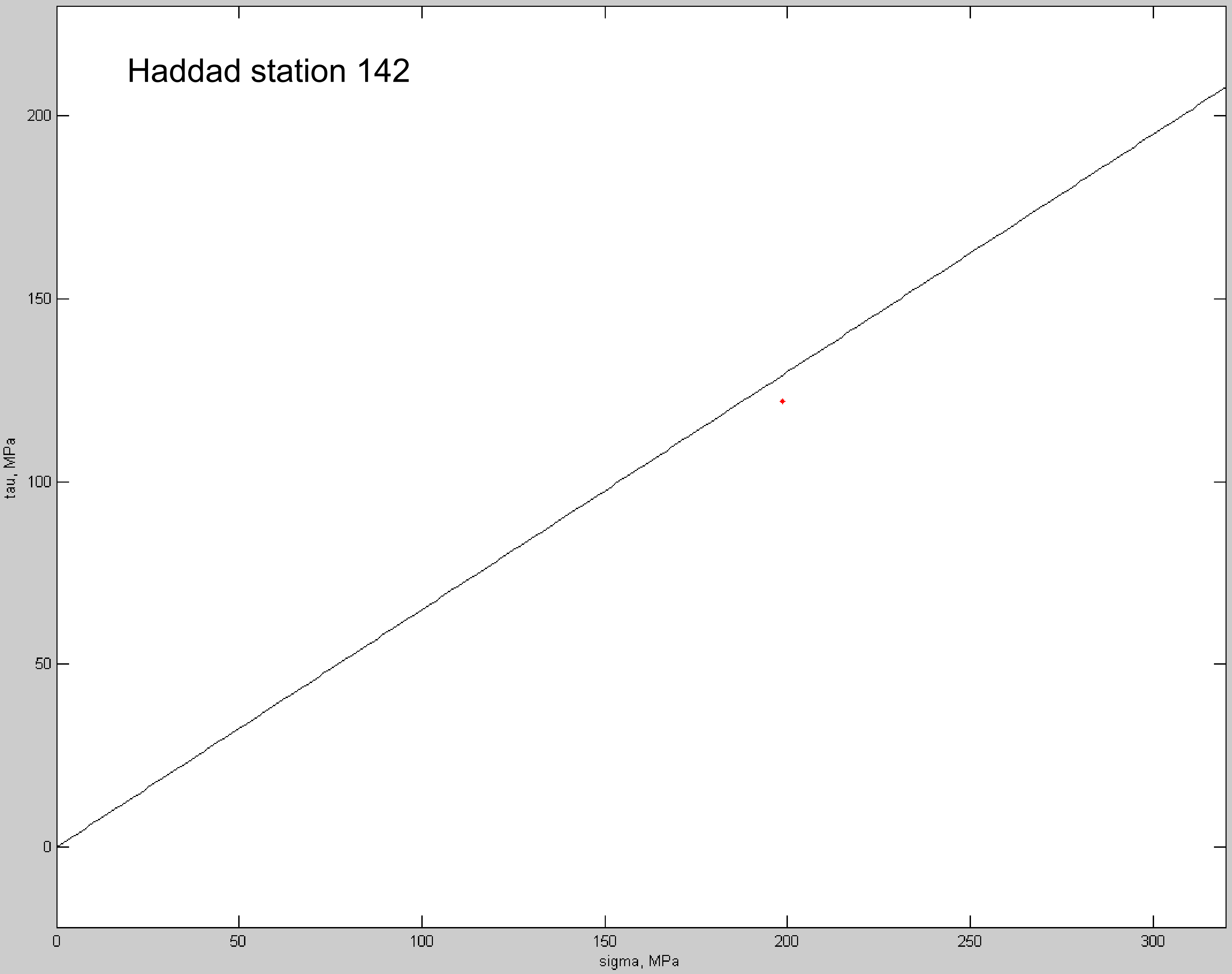

N
sigma3
sigma1
T
σ
τ
B
sigma2
pole
primitive
fault plane
sigma1
sigma2
sigma3
xyz stress1
xyz stress2
xyz stress3
T
B
Ts

%Haddad station 142

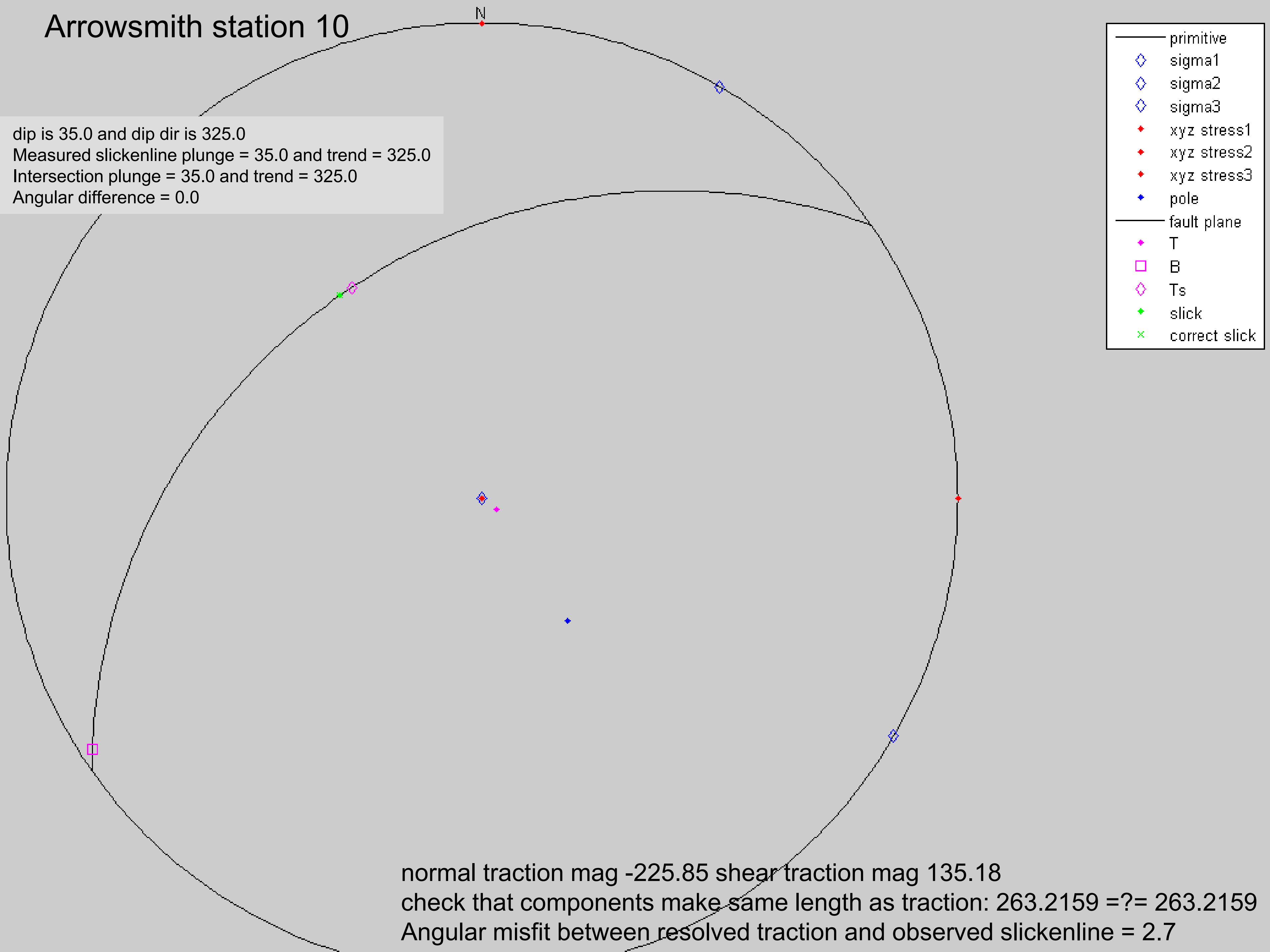
strike = 340;
measured_dip = 32;
slickenlinetrend = 075;
slickenlineplunge = 24;

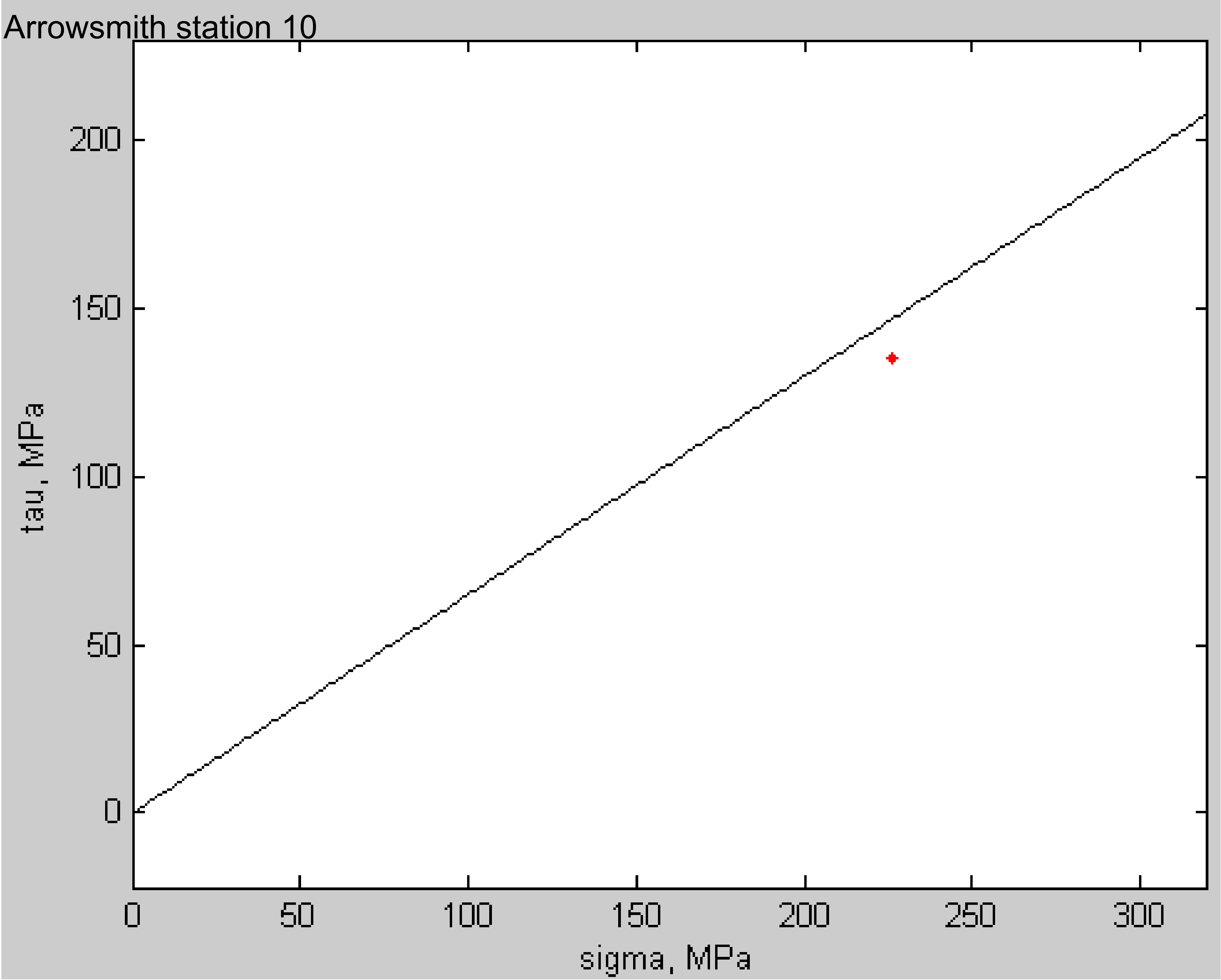


Angular misfit between resolved traction and observed slickenline = 4.7



Haddad station 142
tau, MPa
sigma, MPa
0
50
100
150
200
250
300


Arrowsmith station 10
N
dip is 35.0 and dip dir is 325.0
Measured slickenline plunge = 35.0 and trend = 325.0
Intersection plunge = 35.0 and trend = 325.0
Angular difference = 0.0
primitive
sigma1
sigma2
sigma3
xyz stress1
xyz stress2
xyz stress3
pole
fault plane
T
B
Ts
slick
correct slick
normal traction mag -225.85 shear traction mag 135.18
check that components make same length as traction: 263.2159 =?= 263.2159
Angular misfit between resolved traction and observed slickenline = 2.7



Arrowsmith station 10
tau, MPa
sigma, MPa
200
150
100
50
0
0
50
100
150
200
250
300