Chapter 2

THICKNESS & DEPTH

2.1 DEFINITIONS

Thickness: the perpendicular distance between the parallel planes bounding a tabular body, as displayed on any section perpendicular to these planes; also called the true or stratigraphic thickness (Fig. 2.1).

Apparent Thickness: the distance between the bounding planes measured in some other direction, for example, the perpendicular distance between the traces of the bounding planes on an oblique section, or in some other specified direction, as in a drill hole. It is always greater than true thickness.

Outcrop width: the strike-normal distance between the traces of the parallel bounding planes measured at the earth's surface. It may be measured horizontally or on an incline.

Depth: the vertical distance from a specified level (commonly the earth's surface) downward to a point, line or plane.

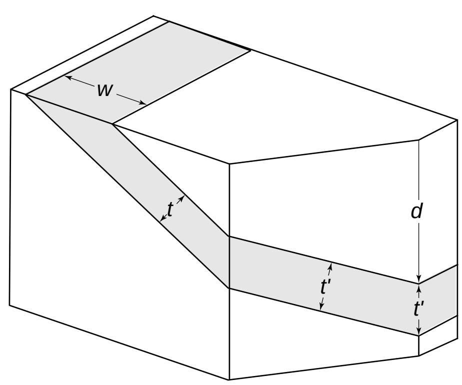

Figure 2.1: True thickness t, apparent thickness t', outcrop width w and depth d.

2.2 THICKNESS DETERMINATIONS

Although geologists may determine the thickness of any stratiform body of rock, most often the concern is with the thickness of layers of sedimentary rocks. In this context "measuring a section" generally refers to a lithologic description of the rock strata as well as a determination of their thicknesses (Kottlowski, 1965;

Compton, 1985). Here, the concern is with thickness alone. The thickness of a layer may be determined in a number of ways. In special circumstances it may be possible to measure it directly, otherwise it must be determined from indirect measurements.

2.3 THICKNESS BY DIRECT MEASUREMENT

Several examples will illustrate how thickness may be measured directly. In a simple case the thickness of a horizontal layer exposed on a vertical cliff face may be obtained by hanging a measuring tape over the edge of the cliff (Fig. 2.2a). Alternatively, if the elevations of the top and bottom of the horizontal layer can be determined accurately, the thickness is simply the difference of the two elevations regardless of slope angle. Another special case involves the exposure of a vertical layer on a horizontal surface; a tape measure extended perpendicular to the strike allows the thickness to be obtained directly (Fig. 2.2b).

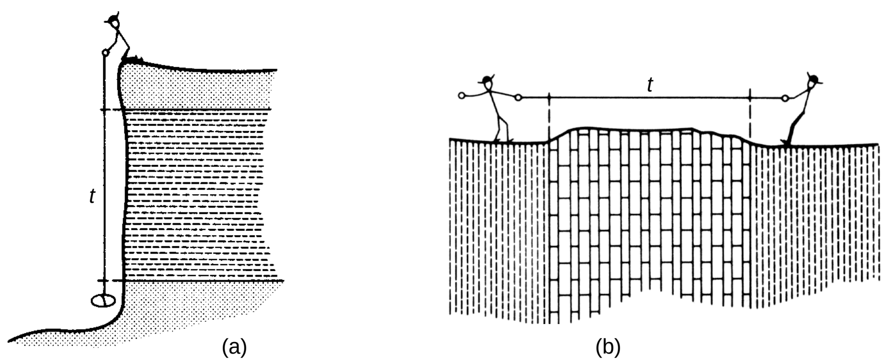

Figure 2.2: Direct measurement of thickness: (a) horizontal layer; (b) vertical layer.

More generally, thickness may be measured directly regardless of the relationship between slope and dip with a Jacob's staff (a light pole with gradations and clinometer or Brunton compass attached at the top; see Robinson, 1959; Hansen, 1960; Freeman, 1991, p. 25). The staff is tilted toward the dip direction through the dip angle (Fig. 2.3a) and a point on the ground is sighted in. The thickness of the layer or portion of the layer between the base of the staff and the sighted point is equal to the length of the staff (Fig. 2.3b). For layers less than staff height the gradations are used, and by occupying successive positions units of any thickness may be measured (Fig. 2.3c).

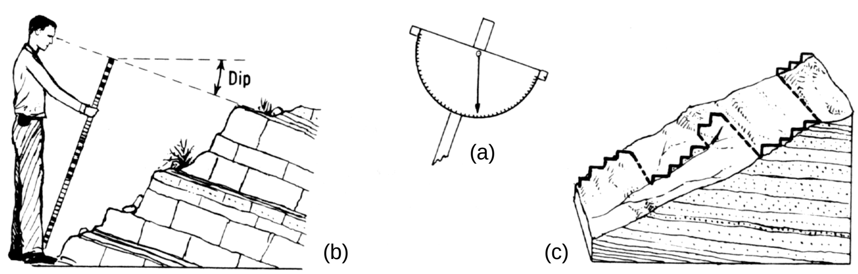

Figure 2.3: Thickness with a Jacob's staff (from Compton, 1985, p 230): (a) simple clinometer; (b) sighting down the dip; (c) stepwise course of measurements.

The principle common to each of these approaches is that if a line of sight can be obtained parallel to the dip direction, the layer appears in edge view, and the true thickness can be obtained by measuring across this view perpendicular to the two parallel bounding planes.

2.4 THICKNESS FROM INDIRECT MEASUREMENTS

When direct measurement of thickness is not possible, there are several alternatives. Which of these is adopted depends on the field situation, on the equipment at hand, on the accuracy required, and finally on personal preference. Given a choice, it is always desirable to make the most nearly direct measurements possible.

All the solutions of true thickness require an edge view of the layer, that is, the image of the layer on a plane perpendicular to bedding. Of the many such planes one can always be readily found or constructed — it is the vertical plane parallel to the line of true dip.

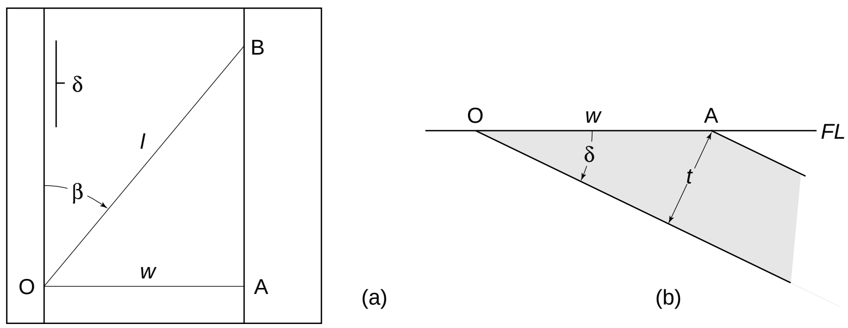

Figure 2.4: Thickness from outcrop width: (a) map; (b) strike-normal section.

The simplest of the indirect approaches is to measure the width of the exposed layer perpendicular to the strike direction on a horizontal plane (OA in Fig. 2.4a). Two measurements are required: the outcrop width w of the layer and the dip angle δ. Then the thickness t can be determined graphically in either of two ways.

1. With the map, a folding line can be used to construct a strike-normal section, a procedure which is virtually identical to that used in problems of dip and strike in Chapter 1.

2. The field measurements can be used to plot the required section directly (Fig. 2.4b).

The thickness may also be calculated from

$$t = w \sin \delta. \tag{2.1}$$

Because of obstructions or lack of exposure it is not always possible to make measurements in the strike-normal direction. For an oblique horizontal traverse (OB in Fig. 2.4a), a correction is required.[1] In effect, the traverse length l is too long and must be reduced to the equivalent outcrop width w. This adjustment can be made with a scaled drawing of the horizontal right triangle OAB. Then just as in the previous case the thickness can be measured on the strike-normal section (Fig. 2.4b). The correction may also be calculated from

$$w = l \sin \beta,$$

where β is the structural bearing of the traverse. A complete analytical solution can be obtained by substituting this result into Eq. 2.1 giving

$$\boxed{t = l \sin \beta \sin \delta} \tag{2.2}$$

In the more general case, thickness is determined from measurements made on sloping ground. We first consider the case where it is possible to measure the outcrop width directly. There are two alternatives:

[1] Note that a vertical section constructed in the direction OB using the apparent dip angle would show the apparent thickness not the true thickness.

1. Thickness can be determined from the slope distance and slope and dip angles along the measured strike-normal traverse.

2. It can also be found from the vertical and horizontal distances between the two ends of the traverse if the slope angle is known.

Each approach has advantages. The first method yields simpler relationships. The second is convenient when highly variable slopes are involved and it can also be used to obtain thickness from measurements made directly on a geologic map.

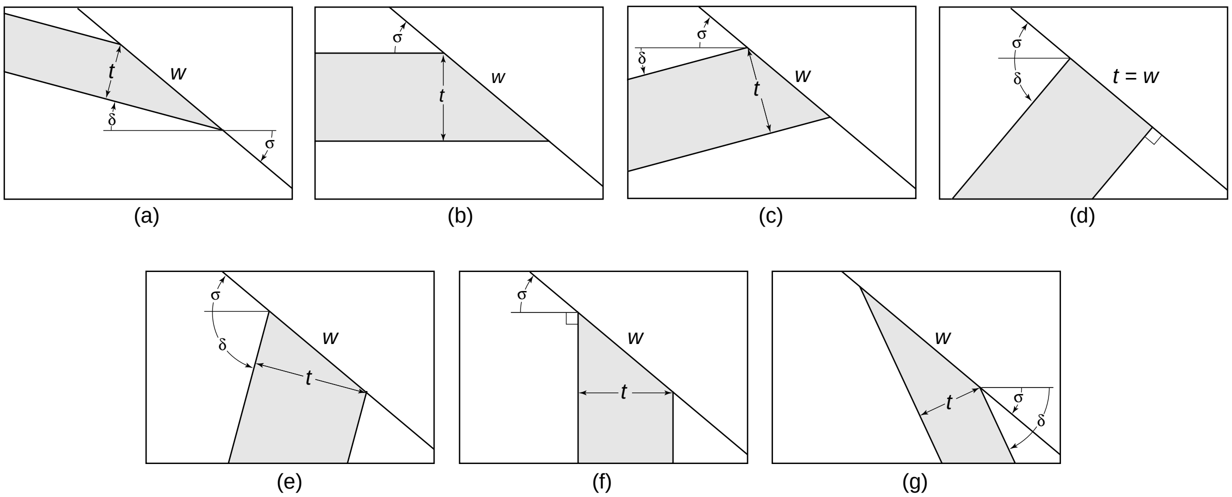

Figure 2.5: Thickness determination from a strike-normal traverse on a slope.

When the outcrop width is measured directly, the approach is closely related to the result of Fig. 2.4, except that thickness is now a function of both dip angle δ and slope angle σ (sigma). There are seven cases, and all are easily solved graphically from a simple scaled cross-section based on the field measurements (see Fig. 2.5). Analytical solutions are also available for all cases.

1. Slope and dip are in the same direction, $\delta < \sigma$ (Fig. 2.5a),

$$t = w \sin(\sigma - \delta). \tag{2.3a}$$

2. The bed is horizontal, $\delta = 0°$ (Fig. 2.5b),

$$t = w \sin \sigma. \tag{2.3b}$$

3. Slope and dip are in the opposite directions, $(\delta + \sigma) < 90$ (Fig. 2.5c),

$$t = w \sin(\delta + \sigma). \tag{2.3c}$$

4. Slope and dip are in the opposite directions, $(\delta + \sigma) = 90$ (Fig. 2.5d),

$$t = w. \tag{2.3d}$$

5. Slope and dip are in the opposite directions, $(\delta + \sigma) > 90$ (Fig. 2.5e),

$$t = w \sin[180 - (\delta + \sigma)] = w \sin(\delta + \sigma). \tag{2.3e}$$

6. The bed is vertical, $\delta = 90$ (Fig. 2.5f),

$$t = w \sin(90 - \sigma) = w \sin(90 + \sigma). \tag{2.3f}$$

7. Slope and dip are in the same direction, $\delta > \sigma$ (Fig. 2.5g),

$$t = w \sin(\delta - \sigma). \tag{2.3g}$$

All these separate cases can be expressed as a single equation by adopting a special sign convention.

1. If the slope and dip are in *opposite* directions the *sum* $(\delta + \sigma)$ is used.
2. If the slope and dip are in the *same* direction the *difference* $(\delta - \sigma)$ or $(\sigma - \delta)$ is used.

The general equation is then

$$\boxed{t = w \sin |\delta \pm \sigma|} \tag{2.4}$$

where, because a negative thickness has no meaning, the absolute value of the angle is used.

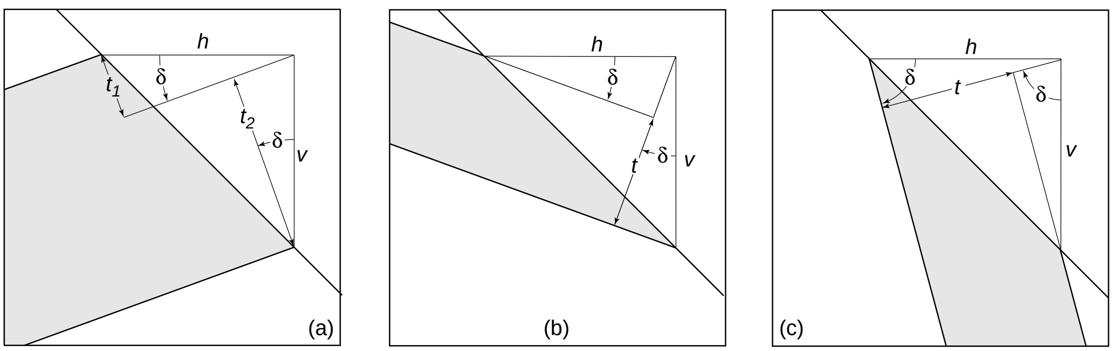

Figure 2.6: Thickness from horizontal h and vertical v components.

The second approach involves determining the horizontal h and vertical v distances between two end points of a strike-normal traverse (Fig. 2.6). Again the stratum may be dipping in the same direction as the slope and or the dip and slope directions may be opposite. In each case it is a simple matter to plot the field data on a scaled vertical section and measure the thickness. Thickness may also be computed. The general approach requires expressions for two partial thicknesses

$$t_1 = h \sin \delta \quad \text{and} \quad t_2 = v \cos \delta.$$

There are two main cases.

1. If the slope and dip are in *opposite* directions then $t = (t_1 + t_2)$ (Fig. 2.6a).
2. If the slope and dip are in the *same* directions the total thickness is the difference of the two partial thicknesses. There are two subcases.
   (a) If $(\delta < \sigma)$ then $t = (t_1 - t_2)$ (Fig. 2.6b).
   (b) If $(\delta > \sigma)$ then $t = (t_2 - t_1)$ (Fig. 2.6c).

Using the same sign convention as before all three cases can then be written as

$$\boxed{t = |h \sin \delta \pm v \cos \delta|} \tag{2.5}$$

The more general case involves an oblique traverse (Fig. 2.7). From the horizontal right-triangle ABD, the horizontal distance h in the strike-normal direction between the two stations is given by

$$\sin\beta = h/h' \quad \text{or} \quad h = h'\sin\beta, \tag{2.6}$$

where h' is the horizontal distance in the oblique direction between the two stations. If a graphical solution is desired first obtain the distance h from the map and then plot the data on a strike-normal section. The full analytical solution is obtained by substituting Eq. 2.6 into Eq. 2.5 to give

$$\boxed{t = |h'\sin\beta\sin\delta \pm v\cos\delta|} \tag{2.7}$$

If the slope length and slope angle, rather than the horizontal and vertical distances, are measured in an oblique direction, it would seem to be a simple matter to introduce a similar correction, but there is no easy way of measuring the appropriate angle in the field ($\angle BAC$ of Fig. 2.7). It is therefore necessary to take a different approach. From the horizontal right-triangle ABD

$$\cos\sigma = l/h' \quad \text{or} \quad h' = l\cos\sigma,$$

where l is the slope length and σ is now the slope angle in the direction of this oblique traverse. Combining this with Eq. 2.6 then gives

$$h = l\cos\sigma\sin\beta.$$

From the vertical right-triangle ACD

$$\sin\sigma = v/l \quad \text{or} \quad v = l\sin\sigma.$$

Using these expressions for h and v in Eq. 2.5 then yields the equation, first derived by Mertie (1922, p. 41),

$$\boxed{t = l|\cos\sigma\sin\beta\sin\delta \pm \sin\sigma\cos\delta|} \tag{2.8}$$

This general equation for stratigraphic thickness is easily applied in the field. By identifying relatively uniform slope segments exposing strata with constant attitude, lay a tape measure along the surface and directly measure l for each lithologic unit. With a compass measure the slope σ, structural bearing β, and dip δ. Computing the thickness of the individual beds using a spreadsheet on a laptop computer is then easy. By occupying successive slope segments one can rapidly construct a full stratigraphic column of the exposed rocks.

2.5 APPARENT THICKNESS

In all the previous cases, the true thickness was derived from a measured apparent thickness. In some situations is necessary to determine the apparent thickness from the true thickness, for example, as displayed on an oblique section.

Problem

- If the true thickness $t = 50$ m and the dip $\delta = 30°$, what will be the apparent thickness t' on a vertical section making an angle $\phi = 40°$ with the dip direction?

Construction

1. In a map view represent the outcrop trace of the lower boundary of the layer by strike line S_1 (Fig. 2.8). From a local origin O on this line draw lines in the true dip direction and the required oblique section line.

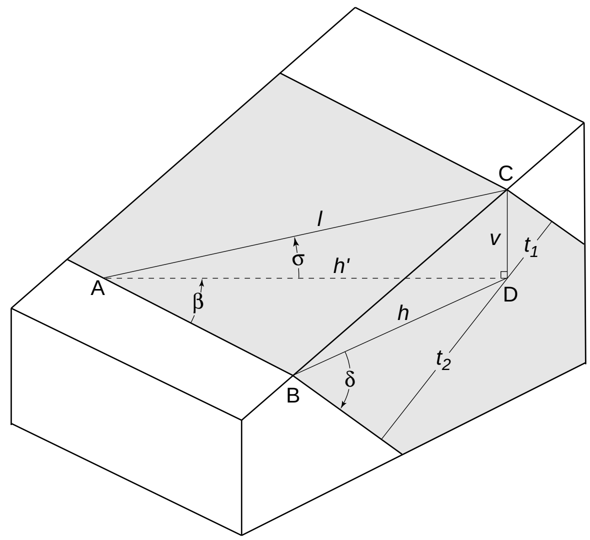

Figure 2.7: Thickness from an oblique traverse on a slope.

2. With the dip line as *FL1* and using $\delta = 30°$ draw the trace of the lower boundary OX. With a convenient scale construct the trace of the upper boundary at a distance $t = 50$ m from OX. This locates point A on the dip line and the outcrop width $w = OA$.

3. On the map draw a second strike line through A to represent the trace of the upper boundary, thus locating point B at the intersection of the oblique section. Now the traverse length $l = OB$.

4. Using Eq. 1.8 find the angle of apparent dip $\alpha = 23.9°$ in this direction. With OB as *FL2* draw the trace of the lower boundary inclined OY at this angle.

5. The perpendicular distance from this inclined trace to point B is the apparent thickness t'.

Answer

- The apparent thickness $t' = 53$ m.

An analytical relationship between true and apparent thickness is also useful (Coates, 1944, p. 7; De Paor, 1987, p. 77; De Paor, 1997, personal communication). From Fig. 2.8 the vertical apparent thickness t'_v, which is the same in triangles OAX and OBY we have

$$t = t'_v \cos\delta \quad \text{and} \quad t = t'_v \cos\alpha.$$

Solving both for t'_v, equating and rearranging gives

$$t' = \left[\frac{\cos\alpha}{\cos\delta}\right] t.$$

Substituting the identities

$$\cos\alpha = 1/\sec\alpha = 1/\sqrt{1+\tan^2\alpha} \quad \text{and} \quad 1/\cos\delta = 1/\sqrt{\cos^2\delta}$$

we then obtain

$$t' = \left[\frac{1}{\sqrt{1+\tan^2\alpha}}\right]\left[\frac{t}{\sqrt{\cos^2\delta}}\right].$$

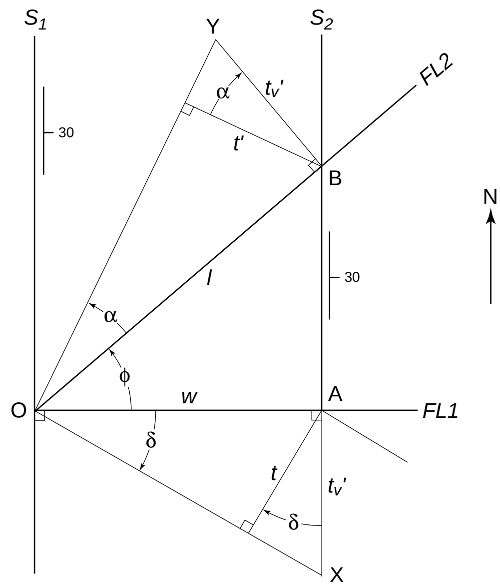

Figure 2.8: Apparent thickness in an oblique section.

From Eq. 1.8

$$\tan\alpha = \tan\delta\cos\phi \quad \text{or} \quad \tan^2\alpha = \tan^2\delta\cos^2\phi,$$

where ϕ is the angle between the true and apparent dip directions. With this the expression for t' becomes

$$t' = \left[\frac{1}{\sqrt{1+\tan^2\delta\cos^2\phi}}\right]\left[\frac{t}{\sqrt{\cos^2\delta}}\right] \quad \text{or} \quad t' = \frac{t}{\sqrt{\cos^2\delta + (\tan^2\delta\cos^2\delta)\cos^2\phi}}.$$

Then with the identities $\tan\delta\cos\delta = \sin\delta$ and $\cos^2\delta = (1-\sin^2\delta)$ this become

$$t' = \frac{t}{\sqrt{\cos^2\delta + \sin^2\delta\cos^2\phi}} = \frac{t}{\sqrt{(1-\sin^2\delta) + \sin^2\delta(1-\sin^2\phi)}}.$$

Expanding and combining terms we finally have

$$\boxed{t' = \frac{t}{\sqrt{1-\sin^2\delta\sin^2\phi}}} \tag{2.9}$$

From the example problem $t = 50$ m, $\delta = 30°$ and $\phi = 40°$, then $t' = 52.8$ m, which is essentially the same result obtained graphically.

2.6 THICKNESS BETWEEN NON-PARALLEL PLANES

Previously the measured layer was taken to be strictly homoclinal, that is, the two bounding planes had identical attitudes. Often, however, the attitudes at the upper and lower ends of a traverse are different. Besides measurement error which we treat later, there are two possible reasons for such divergencies: The bounding planes may not in fact be parallel because the rock body is wedge-shaped rather than tabular, or the layer may be folded.

If the departure from parallelism is small, thickness may be approximated by using the mean of the two dip angles and the mean of the two structural bearings

$$\delta = \tfrac{1}{2}(\delta_1 + \delta_2) \quad \text{and} \quad \beta = \tfrac{1}{2}(\beta_1 + \beta_2) \tag{2.10}$$

in Eqs. 2.2 or 2.8. If the deviation from parallelism is greater, shorter intervals with more nearly parallel boundaries can be treated separately, and the results summed to give an estimate of the total thickness.

If the beds are folded, then the boundaries are curved surfaces rather than planes and the matter is considerably more complicated. If it can be assumed that these bounding surfaces are still parallel, that is, the distance between the two surfaces measured perpendicular to them is constant, then the thickness can be estimated by a simple construction involving tangent arcs (Hewett, 1920).[2]

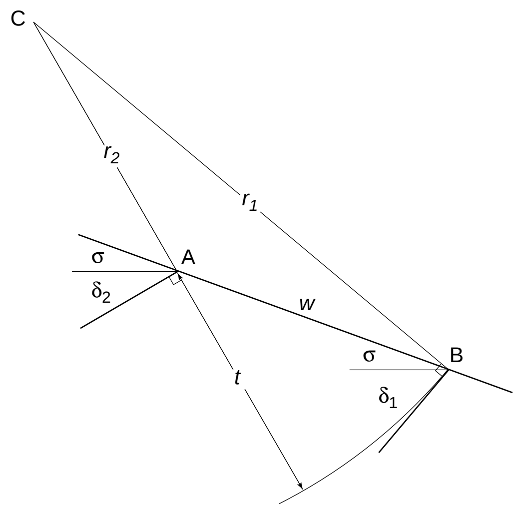

Figure 2.9: Thickness of a folded layer.

Problem

- A strike-normal traverse is made on a slope. The measured strike directions at the upper and lower ends of the traverse are the same, but the dip angles are not. Estimate the thickness of the folded bed.

Construction

1. Draw a scaled cross-section showing the slope angle σ along the traverse line and the two measured dip lines at stations A and B, where the measured slope distance $w = AB$ (Fig. 2.9).
2. At each station construct the dip normals r_1 and r_2 to the dip lines to intersect at common point C.
3. With C as center, draw an arc with radius of BC. The thickness t is the distance between A and this arc measured in the direction of r_2.

The thickness in this case may also be obtained trigonometrically. Labeling the dip angles so that $\delta_1 > \delta_2$ and the corresponding radii $r_1 > r_2$, then by the Law of Sines for the oblique triangle ABC we have

$$\frac{r_1}{\sin A} = \frac{r_2}{\sin B} = \frac{w}{\sin C}.$$

The lengths of the two radii are then

$$r_1 = \frac{w \sin A}{\sin C} \quad \text{and} \quad r_2 = \frac{w \sin B}{\sin C}.$$

[2]Following Busk (1929) we use an extended version of this method in §15.x to reconstruct folds in cross section.

Because the total thickness $t = (r_1 - r_2)$ we have

$$t = w\left[\frac{\sin A - \sin B}{\sin C}\right]. \tag{2.11}$$

With the angular relationships

$$\sin A = \sin\left[90^\circ + (\delta_2 + \sigma)\right] = \cos(\delta_2 + \sigma),$$
$$\sin B = \sin\left[90^\circ - (\delta_1 + \sigma)\right] = \cos(\delta_1 + \sigma),$$
$$\sin C = \sin\left[180^\circ - A - B\right] = \sin(\delta_1 - \delta_2),$$

and Eq. 2.11 becomes

$$\boxed{t = w\left[\frac{\cos(\delta_2 + \sigma) - \cos(\delta_1 + \sigma)}{\sin(\delta_1 - \delta_2)}\right]} \tag{2.12}$$

An important consequence of this construction is that if the dip angles at each end of the traverse are known, all intermediate dips are fixed. The dip at any intermediate point D can be found as the tangent of the concentric arc with C as center and CD as radius.

If the actual dip angles at intermediate points differ then the thickness determination using parallel arcs will be in error. One approach is to treat adjacent pairs of dips separately and sum the incremental thicknesses so determined. Mertie (1940) described the use of parallel curves of a more general nature which takes into account additional dip measurements. This gives a better representation of the thickness of the layers, but constructing these curves is involved and the method is little used.

Another limitation is imposed if the two strike directions differ, a situation which suggests that the fold is not horizontal. True thickness then can no longer be represented in a vertical section. This and other matters related to fold geometry are considered in greater detail in later chapters.

2.7 THICKNESS IN DRILL HOLES

In sub-surface exploration by drilling it is important to determine the thickness of strata from measurements made in the drill holes or in recovered cores. This is especially important in the petroleum industry and Tearpock & Bischke (1991) give a comprehensive treatment.

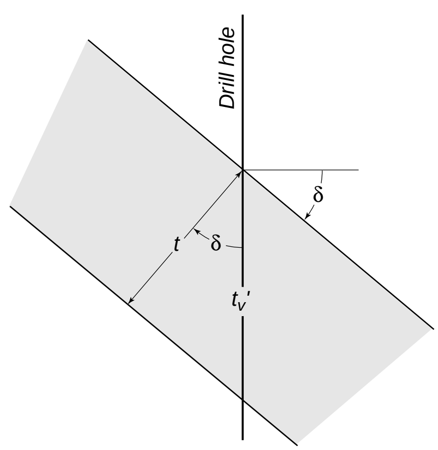

Figure 2.10: Thickness in vertical drill hole.

If the hole is vertical then the determination of the thickness of a layer penetrated by the drill is particularly straight forward. From Fig. 2.10

$$t = t_v' \cos\delta, \tag{2.13}$$

where δ is the dip of the bed and t'_v is the apparent thickness as measured in the vertical drill hole.

Holes which are exactly vertical are difficult to drill, especially if the beds are steeply dipping. The measure of the angular departure of a drill hole from vertical is termed *drift*, measured by the drift angle ψ. There are two cases. If the drift is exactly in the *down-dip* direction (Fig. 2.11a)

$$t = t'_m \cos(\delta + \psi),$$

where t'_m is the measured apparent thickness in the inclined hole. If the hole is exactly in the *up-dip* direction (Fig. 2.11b)

$$t = t'_m \cos|\delta - \psi|.$$

These two can be written as a single equation

$$\boxed{t = t'_m \cos|\delta \pm \psi|} \tag{2.14}$$

where the positive sign is used if the drift has an down-dip component and the negative sign is used if it has a up-dip component.

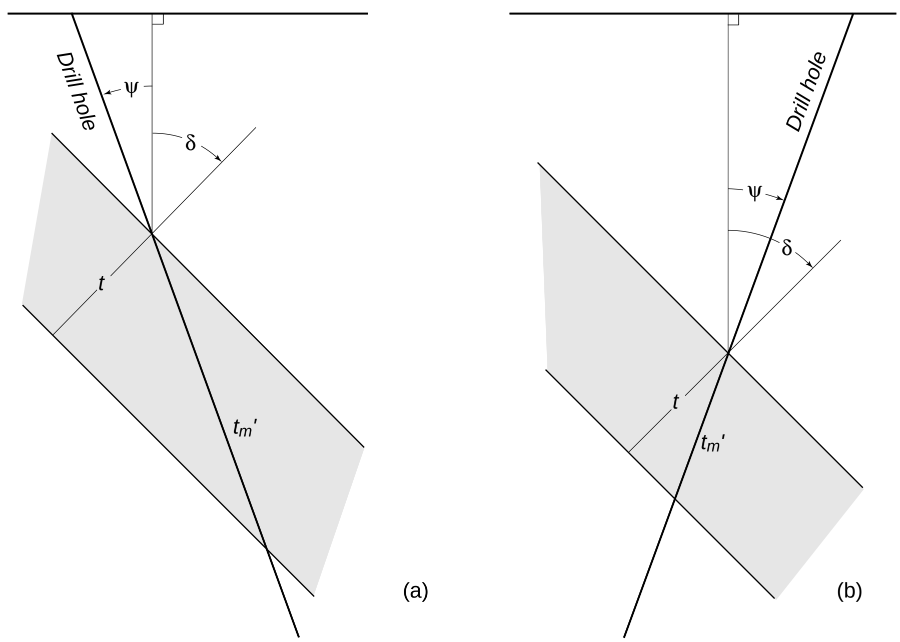

Figure 2.11: Thickness in inclined drill hole: (a) down-dip drift; (b) up-dip drift.

If the drift is oblique to the true dip direction then the apparent dip in the vertical plane containing the drill hole is used giving

$$\boxed{t = t'_m \cos|\psi \pm \alpha|} \tag{2.15}$$

2.8 DEPTH TO A PLANE

Once the relationships involved in the determination of thickness can be visualized problems involving depth should present little additional difficulty for they follow closely same the methods.

As with thickness, the simplest case is the depth to an inclined plane from a horizontal surface at a distance m measured from a point on the outcrop trace of the plane in a strike-normal direction to the

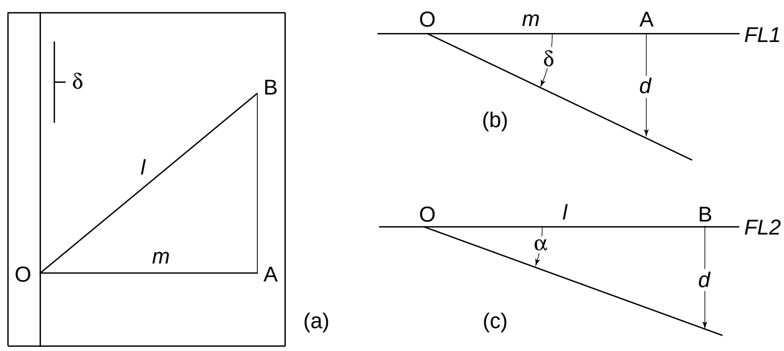

Figure 2.12: Depth: (a) map; (b) strike-normal section; (b) oblique section.

surface point there depth is required. The depth may be found by constructing a scaled triangle, as in the map of Fig. 2.12a, or by using the formula

$$d = m \tan \delta. \tag{2.16}$$

If distance l is measured oblique to the strike, the apparent dip in the traverse direction is used giving

$$d = l \tan \alpha. \tag{2.17}$$

From the previous result of Eq. 1.4

$$\tan \alpha = \sin \beta \tan \delta.$$

Using this is Eq. 2.17 we have an expression for the depth directly in terms of the true dip angle and structural bearing of the drill hole

$$d = l \sin \beta \tan \delta. \tag{2.18}$$

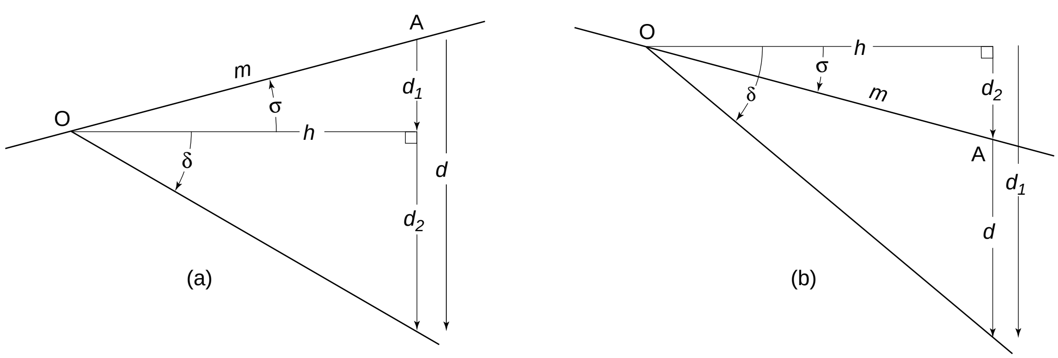

Figure 2.13: Depth to inclined plane: (a) σ and δ in opposite directions; (b) σ and δ in same direction.

The next case involves the depth from a point on a slope. For the case when slope and dip are in opposite directions (Fig. 2.13a)

$$d_1 = h \tan \delta \quad \text{and} \quad d_2 = m \sin \sigma.$$

Because $h = m \cos \sigma$ and the total depth $d = (d_1 + d_2)$ we then have

$$d = m(\cos \sigma \tan \delta + \sin \sigma).$$

If the slope and dip are in the same directions and $(\delta > \sigma)$ (Fig. 2.13b), the total depth $d = (d_1 - d_2)$. Then

$$d = m(\cos \sigma \tan \delta - \sin \sigma).$$

Combining gives

$$\boxed{d = w|\cos\sigma\tan\delta \pm \sin\sigma|} \tag{2.19}$$

If ($\delta < \sigma$) then "depth" is measured upward, as might occur in a mine. This will be signaled by $-d$.

When the measurements are made oblique to the strike, Eq. 2.19 can be written in terms of the traverse length and the apparent dip

$$d = l|\cos\sigma\tan\alpha \pm \sin\sigma|,$$

and with Eq. 1.4 this becomes (after Mertie, 1922, p. 48)

$$\boxed{d = l|\cos\sigma\tan\delta\sin\beta \pm \sin\sigma|} \tag{2.20}$$

2.9 DISTANCE TO A PLANE

A closely related measure is the distance to a plane in a direction other than vertical, as, for example, along an inclined drill hole. This distance may be found graphically by constructing a scaled section, or it may be calculated.

The simpler situation occurs when the trend of the inclined hole is normal to the strike of the plane at a known slope distance from the plane. We first express the depth of the plane below the site of the drill hole (surface point A in Fig. 2.14) by the two partial depths

$$d_1 = h\tan p \quad \text{and} \quad d_2 = h\tan\delta$$

where h is the horizontal projection of the drill hole and p is its plunge angle.

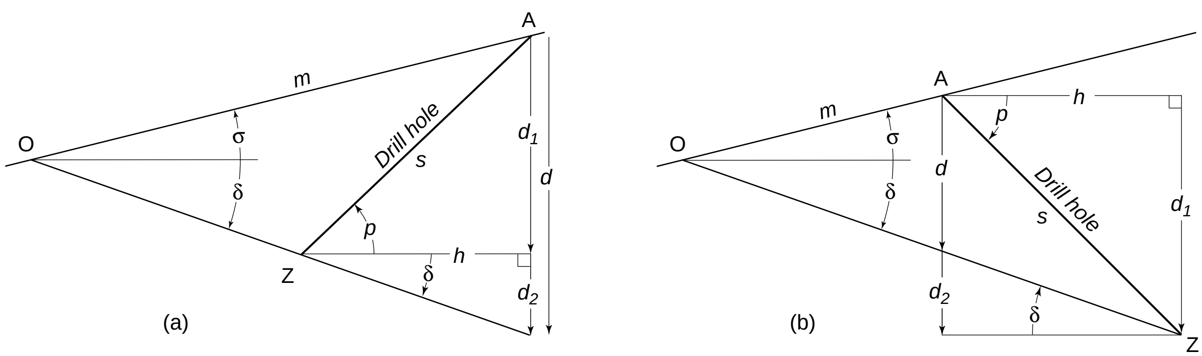

Figure 2.14: Distance in vertical, strike-normal section: (a) δ and p in opposite directions; (b) δ and p in same direction.

There are two cases. In the vertical plane containing the drill hole the plunge and dip may be *opposite* in direction or the plunge and dip may be in the *same* direction.

1. In the first case, from Fig. 2.14a $h = s\cos p$ and $d = d_1 + d_2 = h(\tan p + \tan\delta)$. Combining these, using the identity $\sin p = \cos p\tan p$, and solving for the *inclined distance* s we have

$$s = \frac{d}{\sin p + \cos p\tan\delta}.$$

2. In the second case, from Fig. 2.14b, where $d = d_1 - d_2$, we obtain

$$s = \frac{d}{\sin p - \cos p\tan\delta}.$$

These two expressions can be combined into a single equation using the same sign convention for dip and plunge directions.

$$s = \frac{d}{\sin p \pm \cos p \tan \delta}. \tag{2.21}$$

If the vertical plane containing the drill hole is oblique to the strike, then the apparent dip in this direction can be used giving

$$s = \frac{d}{\sin p \pm \cos p \tan \alpha}$$

or the correction of Eq. 1.4 can be incorporated directly (after Mertie, 1922, p. 48)

$$\boxed{s = \frac{d}{\sin p \pm \cos p \sin \beta \tan \delta}} \tag{2.22}$$

Note that neither the slope angle nor slope length enters into this equation. However, both are accounted for in the expression for d of Eq. 2.20 which, when used in conjunction with Eq. 2.22, gives the required inclined distance.

2.10 ERROR PROPAGATION

As we have seen in §1.4 measured *angles* cannot be absolutely accurate. Similarly, the measured *lengths* are also subject to error. Because of these inevitable errors, it then follows that the calculation of any derived quantity — an angle, a depth, a distance or a thickness — will also be uncertain. In other words, the measurement errors will *propagate* through the calculations. We give a brief introduction to the methods of determining these propagated errors.[3]

The error or uncertainty Δx associated with the measurement of a quantity x is usually expressed in the *standard form*

$$(\text{measured value of } x) = x_{best} \pm \Delta x.$$

This means that the best estimate of the quantity is x_{best} or close to it, and that we can be confident that the correct value *probably* lies between $x_{best} - \Delta x$ and $x_{best} + \Delta x$, though it is *possible* that it lies slightly outside this range.

There are several ways of representing the uncertainty associated with a given measurement (Taylor, 1997, p. 26–29). For example, if measured length $l_{best} = 50$ m has an uncertainty $\Delta l = 2$ m then:

1. The *absolute uncertainty* (or simply *uncertainty*) is expressed in the same units as the measurement itself
$$l_{best} \pm \Delta l = 50 \pm 2 \text{ m}.$$

2. The *fractional uncertainty*, also called the *relative uncertainty* or *precision*, is the dimensionless number
$$\frac{\Delta l}{|l_{best}|} = \frac{1 \text{ m}}{50 \text{ m}} = 0.02,$$
and this gives an estimate of the quality of the measurement.

3. The *percentage uncertainty* is just the fractional uncertainty expressed as a percentage
$$\frac{\Delta l}{|l_{best}|} \times 100\% = 2\%.$$

[3]Taylor (1997) gives the basic theory in an easily accessible form, and we follow his treatment closely. Vacher (2001c, 2001d) treats a number of geological applications.

As a starting point we review some fundamental concepts of differential calculus. Given a function of a single variable

$$y = f(x), \tag{2.23}$$

an infinitesimally small change dx in the independent variable x results in an infinitesimally small change dy in the dependent variable y. These infinitesimal quantities are *differentials*.

A *derivative* is a rate of change. The first derivative $f'(x)$ is the ratio of these infinitesimal changes. The derivative of f with respect to x is written as

$$f'(x) = dy/dx, \tag{2.24a}$$

where the ratio of these two infinitesimals describes the rate of change of y with respect to x; geometrically this is the slope of the curve representing $y = f(x)$. Rearranging, we may write this as

$$\boxed{dy = f'(x)\, dx} \tag{2.24b}$$

which makes the distinction between differentials and the derivative clear. The official definition of a derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If we remove the limit, this equation will hold only approximately, that is,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{2.25a}$$

With $\Delta y = f(x + \Delta x) - f(x)$ we write

$$\boxed{\Delta y \approx f'(x)\, \Delta x} \tag{2.25b}$$

Note the formal similarity of the exact version (Eq. 2.24b) and this approximate version (Eq. 2.25b). We can illustrate the relationship between these two versions graphically.

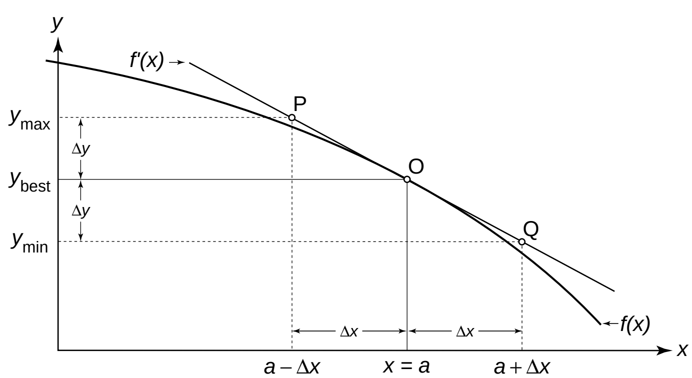

Figure 2.15: Graph of $y = f(x)$ vs. x.

Suppose we measure variable x and that its value is a. This value of x and the corresponding best value of y are represented by the coordinates of point O on the curve representing $f(x)$ (Fig. 2.15). Also suppose that this measured value has an uncertainty of Δx. This results in a propagated uncertainty Δy. We can write this condition as

$$\Delta y \approx \left| \frac{dy}{dx} \right| \Delta x,$$

where the absolute value of the slope is used because the relationship is the same whether it is positive or negative. Points P and Q on the line tangent to the curve at O approximate points on the adjacent curve, and the smaller Δx is, the closer these points will be to this curve.

In our applications it will be useful to adopt the notation which explicitly recognizes that Δy and Δx are errors, as we have in §1.4 (see also Vacher, 2001c, p. 310). Thus we write

$$\varepsilon_y = \left|\frac{dy}{dx}\right| \varepsilon_x. \tag{2.26}$$

Because angles play a prominent role in many structural situations, we start by examining how the uncertainty associated with a single measured angle is propagated.

Problem

- If the measured angle $\theta = 20 \pm 3°$ what is the best estimate of $\cos\theta$ and what is the uncertainty (after Taylor, 1997, p. 65)?

Solution

1. The best estimate is, of course, $\cos\theta = \cos 20 = 0.94$.
2. Then according to Eq. 2.26
$$\varepsilon_{\cos\theta} = \left|\frac{d(\cos\theta)}{d\theta}\right| \varepsilon_\theta = |\sin\theta|\,\varepsilon_\theta \quad (\varepsilon_\theta \text{ in radians}).$$
3. Because ε_θ must be expressed in radians[4] we therefore have $\varepsilon_\theta = 3° = 0.05$ rad.
4. With this value of ϵ_θ we then have $\varepsilon_{\cos\theta} = \sin 20 \times 0.05 = 0.34 \times 0.05 = 0.02$.
5. The range of values of $\cos\theta$ is then 0.94 ± 0.02.

There is a closely related approach to this type of problem (Courant & John, 1965, p. 490–492; see also Vacher, 2001c, 2001d). Any function of a single variable $f(x)$ can be represented by the Taylor series over some interval in the neighborhood of the point $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \tag{2.27}$$

The first term $f(a)$ represents a point on the curve. The second term $f'(a)(x-a)$ represents the slope of the tangent line through this point. Each of the higher order terms represents a curve which bring the sum into closer correspondence with $f(x)$. If $\Delta x = (x-a)$ is small, then powers of Δx will be much smaller and can be neglected, and we are left with the approximation

$$\Delta y \approx f'(a)\Delta x$$

which is the same as Eq. 2.25b.

Measurement errors which are propagated to trigonometric functions are, of course, also propagated to calculations which use these functions.

Problem

- How will the uncertainty in the measurement of the angle of dip $\varepsilon_\delta = 1°$ influence the calculation of the depth to a plane?

[4] Multiply by $\pi/180 = 0.01745\ldots$ to convert degrees to radians.

δ	d	ε_d	ε_δ/d
10	17.6	±1.8	10%
30	58.0	±2.3	4.0%
45	100	±3.5	3.5%
60	173	±7	4.0%
80	567	±58	10%

Table 2.1: Calculated depth for $\varepsilon_\delta = 1°$.

Solution

1. From the vertical section containing the true dip angle (Fig. 2.16b)
$$d = m \tan\delta. \tag{2.28}$$
Assume that the horizontal distance $m = 100$ m exactly.
2. From Eq. 2.26
$$\frac{d(\tan\delta)}{d\delta} = \sec^2\delta = \frac{1}{\cos^2\delta}$$
3. Then
$$d = m\left[\tan\delta \pm \frac{\varepsilon_\delta}{\cos^2\delta}\right]. \tag{2.29}$$
4. The fractional uncertainty is given by
$$\frac{\varepsilon_d}{d} = \frac{1}{\tan\delta\cos^2\delta}.$$
5. The results for a range of values of δ are shown in Table 2.1.

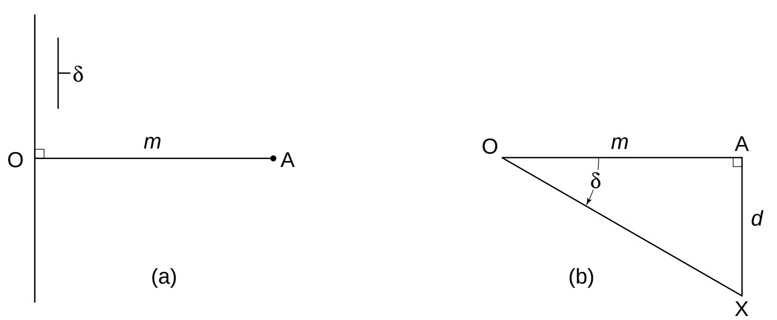

Figure 2.16: Depth to a plane (after Vacher, 2001, p. 312): (a) map; (b) dip section.

For problems involving multiple variables there are several simple rules for the arithmetic involved (Taylor, 1997, p. 49–53; Vacher, 2001d, p. 390–392). For the case of two variable these are

1. If two quantities x and y are measured with uncertainties Δx and Δy and the measured values are used to compute
$$q = x + y \quad \text{or} \quad q = x - y,$$
then the uncertainty in the computed value of q is the sum of the original uncertainties
$$\Delta q \approx \Delta x + \Delta y.$$

2. If two quantities x and y are measured with uncertainties Δx and Δy and the measured values are used to compute
$$q = xy \quad \text{or} \quad q = x/y,$$
then the uncertainty in the value of q is the sum of the original fractional uncertainties
$$\frac{\Delta q}{|q|} \approx \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|}.$$

To summarize, this means that the uncertainty in $q(x, y)$ is
$$\Delta q \approx \left|\frac{\partial q}{\partial x}\right| \Delta x + \left|\frac{\partial q}{\partial y}\right| \Delta y, \tag{2.30}$$
which is just the first-derivative term in the Taylor series for two variables.

Problem

- What will the uncertainty in the depth be if the uncertainty in the dip is $\delta \pm \varepsilon_\delta = 60 \pm 1°$ and the distance $m \pm \varepsilon_m = 100 \pm 2$ m.

Solution

1. Adapting Eq. 2.30 we have
$$\varepsilon_d = \left|\frac{\partial d}{\partial m}\right| \varepsilon_m + \left|\frac{\partial d}{\partial \delta}\right| \varepsilon_\delta.$$
2. Performing the partial differentiations on Eq. 2.28 yields
$$\frac{\partial d}{\partial m} = \tan\delta \quad \text{and} \quad \frac{\partial d}{\partial \delta} = \frac{m}{\cos^2\delta}.$$
3. Then the uncertainty in the depth is
$$\varepsilon_d \approx \varepsilon_m \tan\delta + \varepsilon_\delta \frac{m}{\cos^2\delta} = 2(\tan 60) + \frac{\pi}{180}\left[\frac{100}{\cos^2 60}\right] = 11 \text{ m}.$$

This pattern of combining uncertainties is easily extended to errors associated with more than two measurements. For three variables the formula is
$$\varepsilon_f = \left|\frac{\partial f}{\partial x}\right| \varepsilon_x + \left|\frac{\partial f}{\partial y}\right| \varepsilon_y + \left|\frac{\partial f}{\partial z}\right| \varepsilon_z. \tag{2.31}$$

Problems

1. If $l = 125$ m, $\delta = 22°$ and $\beta = 15°$ what is the depth d to an inclined plane at B (Fig. 2.17)?
2. Now suppose that the uncertainty associated with each of these measures is $l = 125 \pm 3$ m, $\delta = 22 \pm 3°$ and $\beta = 15 \pm 1°$, what now can be say about the depth d (after Vacher, 2001d, p. 394–395)?

Solution 1

1. From the map view (Fig. 2.17a) and the dip section (Fig. 2.17b) we have expressions
$$m = l \sin\beta \quad \text{and} \quad d = m \tan\delta.$$
2. Combining these gives
$$d = l \sin\beta \tan\delta, \tag{2.32}$$
and we can calculate the depth using $l = 125$ m, $\beta = 15°$ and $\delta = 22°$. The answer is $d = 13.1$ m.

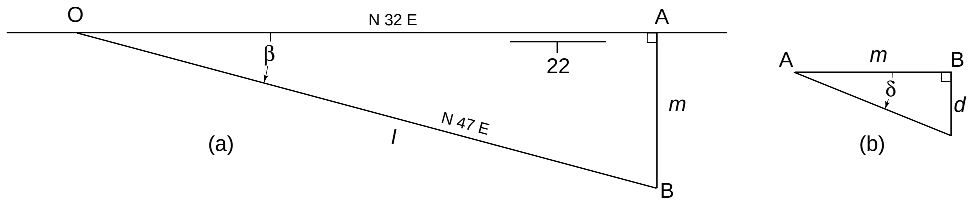

Figure 2.17: Angles and distances (after Vacher, 2001d, p. 394): (a) map; (b) true dip section.

Solution 2

1. From Eq. 2.31 the propagated error ε_d is given by

$$\varepsilon_d = \left|\frac{\partial d}{\partial l}\right| \varepsilon_l + \left|\frac{\partial d}{\partial \beta}\right| \varepsilon_\beta + \left|\frac{\partial d}{\partial \delta}\right| \varepsilon_\delta. \tag{2.33}$$

2. Using Eq. 2.32, form the partial derivatives and plug in values of l, β and δ

$$\frac{\partial d}{\partial l} = \sin\beta \tan\delta = 0.10457, \quad \frac{\partial d}{\partial \beta} = l\cos\beta\tan\delta = 48.78642, \quad \frac{\partial d}{\partial \delta} = \frac{l\sin\beta}{\cos^2\delta} = 37.63349.$$

3. Using these values in Eq. 2.33, together with

$$\varepsilon_l = 2 \text{ m}, \quad \varepsilon_\beta = 0.01349 \text{ rad}(2^\circ), \quad \varepsilon_\delta = 0.05236 \text{ rad}(3^\circ).$$

   we have the result $\varepsilon_d = 3.8$ m. Thus the depth and the total uncertainty is 13.1 ± 3.8 m. Thus the minimum and maximun depths are

$$d_{min} = 13.1 - 3.8 = 9.3 \text{ m} \quad \text{and} \quad d_{max} = 13.1 + 3.8 = 16.9 \text{ m}.$$

If the uncertainties are *not* small, then the second-derivative term in the Taylor series can be used. Vacher (2001d, p. 394–395) gives an example problem and the necessary calculations. If the uncertainties are *independent* and *random* then Eq. 2.30 is likely to overestimates the total uncertainty.[5]

If the measurements are independent then there is a 50% chance that an underestimate of x will be accompanied by an overestimate of y, or vice versa. In such a case, the probability of underestimating or overestimating both x and y by the full amounts Δx and Δy is small, and therefore Δq overstates the probable total error.

Is there a better estimate of Δq? Small measurement errors subject to random uncertainties are described by the Gaussian or normal distribution. If both x and y are measured independently then the uncertainty is given by

$$\Delta q = \sqrt{(\Delta x)^2 + (\Delta y)^2}. \tag{2.34}$$

This is called the *sum in quadrature* (Taylor, 1997, p. 57–62, 141–143) and it is widely used for calculating the uncertainty associated with measurements made in the laboratory by physicists and chemists.

On the other hand, Vacher (2001d, p. 396) argues that because the uncertainties associated with measurements made by geologists in the field may not be small, a better approach is to incorporate the second-derivative term of the Taylor series in calculating uncertainties.

[5]As an example of measurements which are not independent consider a steel tape designed for use at one temperature and used at a different temperature. Under these circumstances all measurements will either be underestimated or overestimated.

2.11 EXERCISES

1. The attitude of a sandstone unit is N 65 E, 35 N. A horizontal traverse with a bearing of N 10 E made from the bottom to the top measured 125 m. Determine the thickness graphically and check your result using Eq. 2.3.

2. The following information is from a geologic map. The attitude of a basalt sill is N 5 W, 38 W. An eastern point on the lower contact has an elevation of 900 m, and a western point on the upper contact has an elevation of 1025 m. The line connecting these two points has a bearing of N 85 W. Determine the thickness of the sill graphically and check your result using Eq. 2.7.

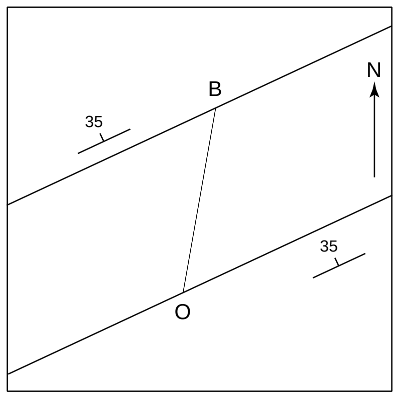

Figure 2.18: Geologic sketch map.

3. The geologic sketch map of Fig. 2.15 shows a thick shale formation between two limestone units exposed on a south-facing slope. A trail angles up brushy slope in a N 30 E direction at a nearly constant 20° angle; the traverse length in crossing the entire shale unit is 366 m. The beds have a consistent attitude of N 80 E, 35 N. If the shale-limestone contact lines are approximately horizontal how steep is the shale slope? What is the difference in elevation between the beginning and ending of the traverse? What is the thickness of the shale?

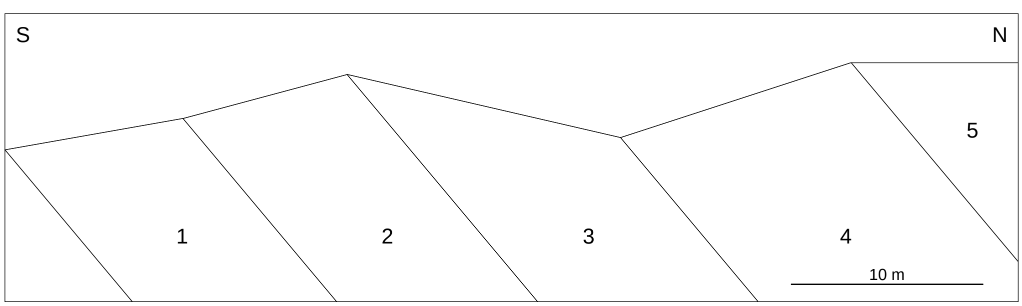

Figure 2.19: Cross section.

4. A south-to-north, strike-normal traverse made across a series of badland beds dipping 50° due north yielded the following data (the setting is shown in Fig. 2.16). Determine the total thickness.

5. A mineralized vein with an attitude of N 37 W, 50 SW is exposed on a ridge crest. How far down a 22° slope in a N 82 W direction would it be necessary to go to find a point at which the vein lies at a depth of 100 m? At that point, what is the minimum inclination and length of a shaft to reach the vein?

Unit	Lithology	slope distance	slope angle
5	upper sandstone	6.7 m	0°
4	upper purple mudstone	17.7 m	18°
3	lower sandstone	8.8 m	−13°
2	pink claystone	8.0 m	15°
1	lower purple mudstone	8.2 m	10°

Table 2.2: Total thickness from field measurements.

6. A 125 m long strike-normal traverse made up a 15° slope between the bottom and top of a limestone strata gave the following information: the dip at the bottom of the unit is 55° and at the top it is 65°, both in a downslope direction. The strike directions at both points are the same. Using the method of tangent arcs estimate the thickness of the unit; check your result with Eq. 2.11. Compare this with the result obtained from Eq. 2.10 using the mean of the two dip angles.