Chapter 6

ROTATIONS

6.1 INTRODUCTION

In a number of geologic situations structural lines and planes have been rotated from some initial orientation and these can be analyzed on the stereonet. Every *rigid rotation* can be defined by an angle and sense of rotation about a specified axis.

The most general case involves rotation about an inclined axis but we start with the simpler cases of rotations about vertical and horizontal axes. We do this because it is a good way to introduce the techniques of rotations and because a sequence of such rotations is equivalent to a rotation about a single inclined axis. In all cases, the sense of rotation is described as clockwise or anticlockwise when looking along the specified axis, whether horizontal, inclined or vertical.

6.2 BASIC TECHNIQUES

As an aid to visualization consider a turntable (Fig. 6.1). As the base rotates about its axis R through some angle ω the locus of an oblique line L through its center O is a right-circular *cone of rotation.* Angle ϕ between R and L is the semi-vertex angle of this cone. The intersection of this cone with the sphere will, in general, be a *small circle*, one in the lower and one in the upper hemisphere. There are, however, two special cases: if $\phi = 0°$ the surface degenerates to a line and if $\phi = 90°$ it becomes a plane.

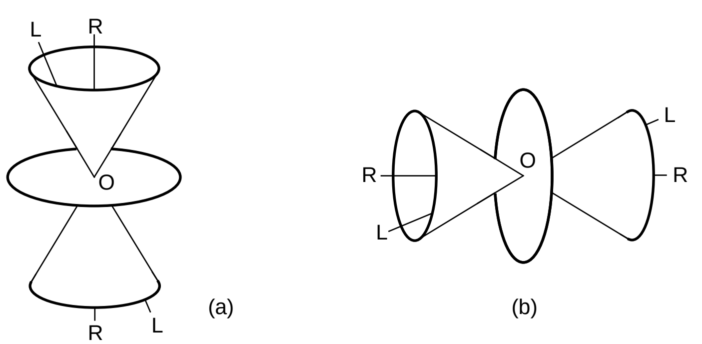

Figure 6.1: Cone of rotation: vertical axis; (b) horizontal axis.

Rotation about a vertical axis is the easiest to perform on the stereonet. To visualize imagine the turntable with its vertical axis downward as in Fig. 6.1a. The cone of rotation intersects the lower hemisphere as a small circle at the center of the net and the sense of rotation can be immediately and directly seen as either *clockwise* or *anticlockwise*.

Problem

- What is the orientation of the horizontal line $L_1(00/150)$ after a anticlockwise rotation $\omega = 70°$ about a vertical axis?

Construction

1. Mark R at the center of the net and plot point L_1 on the primitive representing the line (Fig. 6.2a).
2. From L_1 count off $\omega = 70°$ anticlockwise along the primitive to locate the point L_1' representing the rotated line.

Answer

- After rotation the orientation of the line is $L_1'(00/080)$.

Because $\phi = 90°$, in this special case the trace of the cone of rotation is a great circle. Note too that the trend changed but the line remained horizontal. The construction is only slightly more involved if an inclined line is rotated about vertical axis.

Problem

- What is the orientation of inclined line $L_2(30/150)$ after a anticlockwise rotation $\omega = 70°$ about a vertical axis?

Construction

1. Mark R at the center of the net and plot point L_2 representing the inclined line (Fig. 6.2b).
2. From the trend of L_2 count off $\omega = 70°$ anticlockwise along the primitive to locate the trend of the rotated line. With the original plunge angle plot L_2' representing the rotated line.

Answer

- After rotation the orientation of the line is $L_2'(30/080)$.

Again, note that the trend changed but the plunge remained the same. Both L_2 and L_2' lie on a small circle which represents the intersection of the vertical cone of revolution and the lower hemisphere. As a visual aid, this circle may be added to the stereogram; with a compass draw a circle about the center of the net with angular radius $\phi = (90° - p)$; in this example $\phi = 60°$.

Rotation about a horizontal axis can also be performed readily on the stereonet (Fig. 6.1b). Unlike the case of the vertical axis, there are many possible horizontal axes. Rotations about such axes are always performed with R on the overlay coincident with the north or south point to take advantage of the *small circles* printed on the net. We illustrate with two examples.

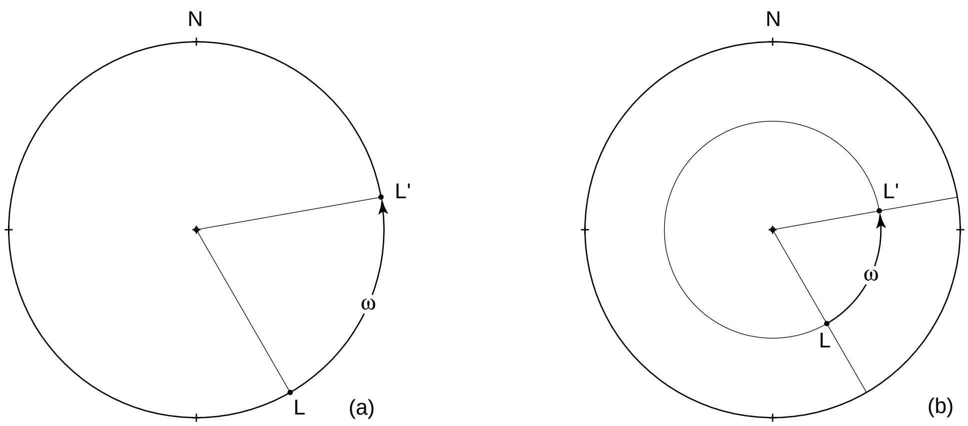

Figure 6.2: Rotations about a vertical axis.

Problem

- What is the attitude of the horizontal line $L_1(00/030)$ after a 110° *clockwise* rotation about a horizontal axis which trends due north?

Visualization

- Looking north, a clockwise rotation about R moves L from right to left along its small circle.

Construction

1. Mark $R(00/000)$ representing the rotation axis and plot point $L_1(00/030)$ representing the horizontal line (Fig. 6.3a).
2. Along the small circle on which L_1 lies count off $\omega = 110°$ from right to left to locate point L_1'.

Answer

- The attitude of the line after rotation is $L_1'(28/349)$.

Note that both the trend and plunge of the line changed as the result of this rotation. The second example involves the more general case of a rotation of an initially inclined line.

Problem

- What is the attitude of line $L_2(40/120)$ after the same rotation $\omega = 110°$?

Construction

1. Again mark $R(00/000)$ and plot point $L_2(40/120)$.
2. Along the small circle on which L_2 lies count off 110° from right to left to locate point L_2' (Fig. 6.3a).

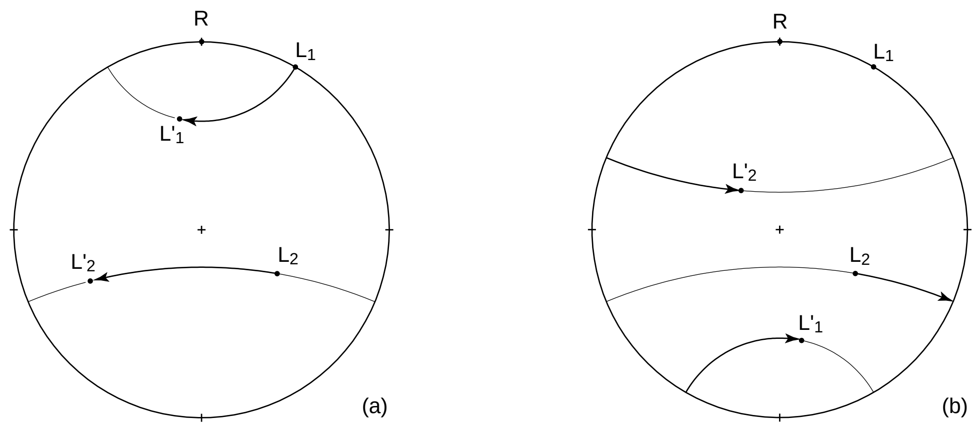

Figure 6.3: Rotations about a horizontal axis: (a) clockwise; (b) anticlockwise.

Answer

- The attitude of the rotated line is $L_2'(24/245)$.

In both these examples the lines remained in the lower hemisphere. Every such structural line has another end, called its *opposite*, which intersects the upper hemisphere and therefore normally remains out of sight. With other senses and angles of rotation, however, the initially downward end may move into the upper hemisphere. When this happens its opposite immediately move into the lower hemisphere. Two closely related examples will illustrate the treatment (Fig. 6.3b).

1. If the horizontal line $L_1(00/030)$ is rotated $\omega = 110°$ *anticlockwise* instead, it immediately moves into the upper hemisphere. At the same instant its opposite moves into the lower hemisphere diametrically opposite and thereafter along the same small circle. The final attitude is $L_1'(28/169)$.

2. If the plunging line $L_2(40/120)$ is similarly rotated $\omega = 110°$ anticlockwise it moves first along its small circle 44° to the primitive and then its opposite continues the rotation along the same small circle an additional 66°. The total rotation is thus made up of two parts. The final attitude is $L_2'(57/315)$.

In addition to lines, we can also rotate planes and there are two ways of doing this. First, several points along the great circle trace of the plane may be rotated individually to establish the great circle representation of the rotated plane.

Problem

- Rotate the plane N 18 W, 50 W clockwise $\omega = 40°$ about a horizontal axis which trends N 30 E.

Method I

1. Trace in the great circle representing the given plane (Fig. 6.4a).

2. Mark $R(00/030)$ and revolve the overlay so that this point coincides with north on the net.

3. In this position, arbitrarily locate three points L_1, L_2 and L_3 on the arc of the great circle. These should be widely spaced and it simplifies things if each is located at the intersection of a small circle.

4. Without moving the overlay count off $\omega = 40°$ from each of these points along the small circles in the direction given by the sense of rotation, that is, from right to left. Doing this for L_1 and L_2 locates the rotated points L'_1 and L'_2 directly. Point L_3, however, is carried to the primitive and beyond, which means that it moves into the upper hemisphere and its opposite L'_3 into the lower hemisphere.

5. Revolve the overlay so that L'_1, L'_2 and L'_3 lie on a great circle which can then be traced in.

Answer

- The strike and dip of the plane after rotation is N 62 W, 35 S. Note that two points would have sufficed to fix the great circle but the third serves as an important check.

With the second method, the plane is represented by its pole and this is rotated in a single step.

Method II

1. Mark $R(00/030)$ and plot the pole of the plane $P(40/072)$ (Fig. 6.4b).
2. Revolve R to north and count off $\omega = 40°$ from P along its small circle to locate the rotated pole P'.
3. Trace in the corresponding great circle representing the plane and read its attitude.

Answer

- The attitude of the pole of the rotated plane is $P'(55/028)$ and this is the same attitude as before.

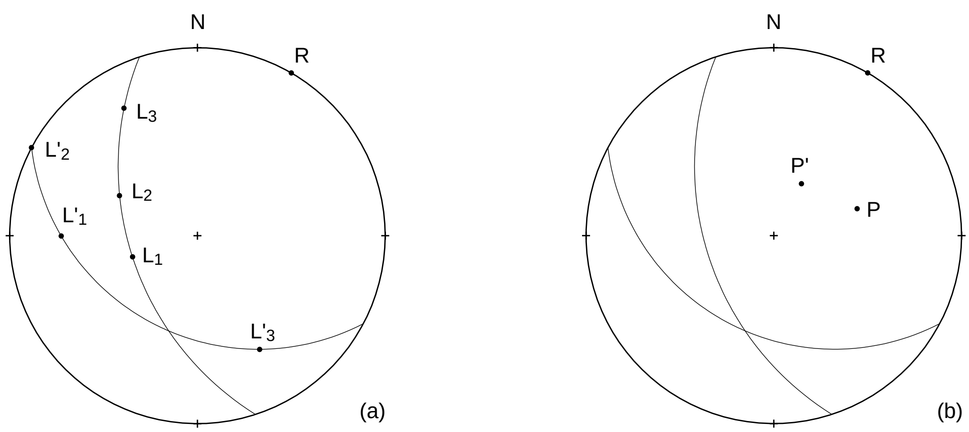

Figure 6.4: Rotation of a plane: (a) points on plane; (b) poles.

Clearly, it is far easier to treat the single point representing the pole of the plane rather than several points on the great circle, although this requires that the dip and strike of the plane be converted to plunge and trend of the pole. Bengtson (1983) described an alternative plot which avoids even this step.

As emphasized by Phillips (1971, p. 5) these rotational techniques involve two closely related but geometrically different and distinct manipulations. The procedure of turning the overlay about its center to align a horizontal axis with north or south on the net is one of convenience, but the overlay always carries with it the N mark so that the original orientations never really changed. The term *revolve* is used specifically to describe this manoeuvre. On the other hand, as the result of a *rotation* lines and planes have entirely new orientations relative to a fixed geographical direction.

6.3 SEQUENTIAL ROTATIONS

A line of any initial orientation can be rotated into any final orientation by a sequence of these simple rotations. To illustrate we rotate an initially horizontal plane, represented by its vertical pole, containing a line in two steps: first about horizontal axis R_H, then about vertical axis R_V.

Problem

- A horizontal plane contains line $L(00/320)$. What is the attitude of the plane and line after a two-step rotation:

    1. Rotate first about axis R_H which trends due south by a clockwise angle $\omega_H = 60°$,
    2. Then rotate about axis R_V by an anticlockwise angle $\omega_V = 40°$.

Visualization

- Looking due south in the direction of R_H we see that a *clockwise* rotation moves pole P and line L to the east (Fig. 6.5a). Alternatively, looking due north in the direction of the opposite of R_H, we see that this is equivalent to an *anticlockwise* rotation. This illustrates a general rule: a clockwise rotation about an axis produces the same results as an anticlockwise rotation about its opposite, and vice verse.

Construction

1. On the primitive, mark points $R_H(00/180)$ and $L(00/320)$, and at the center the coincident points $P(90/000)$ and $R_V(90/000)$ (Fig. 6.5a).
2. There are two ways of performing the first rotation.
    (a) Turn R_H to north. As the horizontal plane tilts clockwise about this axis, its pole P moves 60° to the left from the center along the east-west diameter of the net to P', and line L moves 60° in the same sense along its small circle to L'.
    (b) Leave R_H at the south point; its opposite is now at north. An anticlockwise tilt moves the pole and the line 60° to the right from the center along the east-west diameter of the net to P' and L by the same amount and sense to L'.
3. The second rotation about R_V changes the trends of both the once rotated pole P' and line L' by a anticlockwise angle of 40°, but their inclinations remain the same.

Answer

- After two rotations, the attitude of the pole is $P''(30/050)$ and the attitude of the line is $L''(34/297)$. The corresponding dip and strike of the plane is N 40 W, 60 W and the line trends toward N 63 W.

If the order is reversed, that is, the rotation about R_V by $\omega_V = 40°$ is performed before the rotation about R_H by $\omega_H = 60°$, the result is different (Fig. 6.5b). Because a rotation about a vertical axis does not change the orientation of a vertical line, the pole has been, in effect, only rotated once. Its final attitude is $P''(30/090)$. The line has been rotated twice and its orientation is $L''(59/289)$. This demonstrates that in finite rotations the order of the steps is important, that is, they are not generally *commutative*.

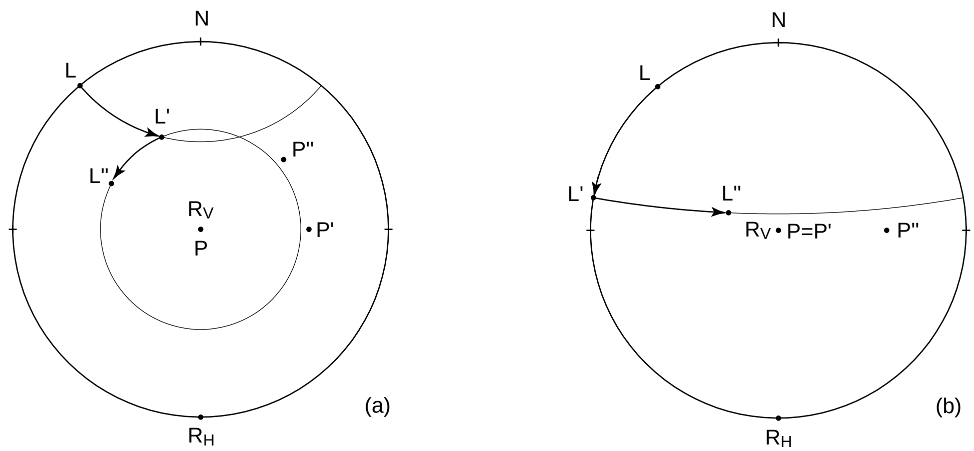

Figure 6.5: Two rotations: (a) about R_H then R_V; (b) about R_V then R_H.

6.4 ROTATIONS ABOUT INCLINED AXES

The general case involves a rotation about an inclined axis. As before, the locus of a line rotated about such an axis is a small circle on the stereonet. We start with the basic geometry of the case in order to establish a visual picture of the process.

Problem

- Rotate the horizontal line $L(00/050)$ about the inclined axis $R(30/090)$ anticlockwise $\omega = 90°$.

Construction

1. Plot inclined axis R and horizontal line L. The angle between these two points measured along the common great circle is $\phi = 48°$ (Fig. 6.6).
2. About R draw the small circle representing the cone of rotation with angular radius ϕ (see §6.9).
3. As L rotates 90° about R it moves along this small circle to $L'(63/033)$.

Besides requiring the extra effort of constructing this small circle, there is, unfortunately, no direct way of measuring the angle of rotation on it. This is not, therefore, a practical approach to performing rotations graphically. The diagram is, however, an important aid to visualizing the effects of a rotation about such an inclined axis. In practice, there are two alternative constructions.

The first depends on previous methods and consists of rotating the inclined axis R about a horizontal axis so that it is either horizontal or vertical. The advantage of adopting a vertical axis is that rotations into the upper hemisphere are commonly avoided. Then the require rotation is performed as in the previous examples. Finally, R is returned to its original inclination by reversing the first rotation.

Problem

- Rotate the horizontal line $L(00/050)$ anticlockwise $\omega = 90°$ about the inclined axis $R(30/090)$.

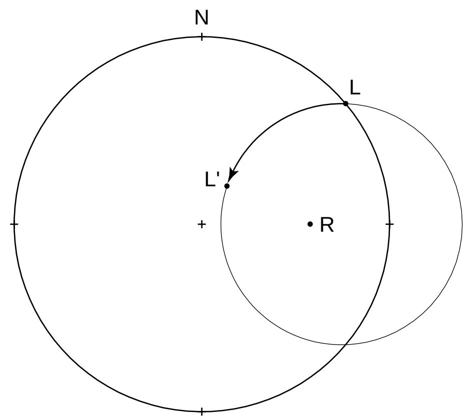

Figure 6.6: Rotation about an inclined axis.

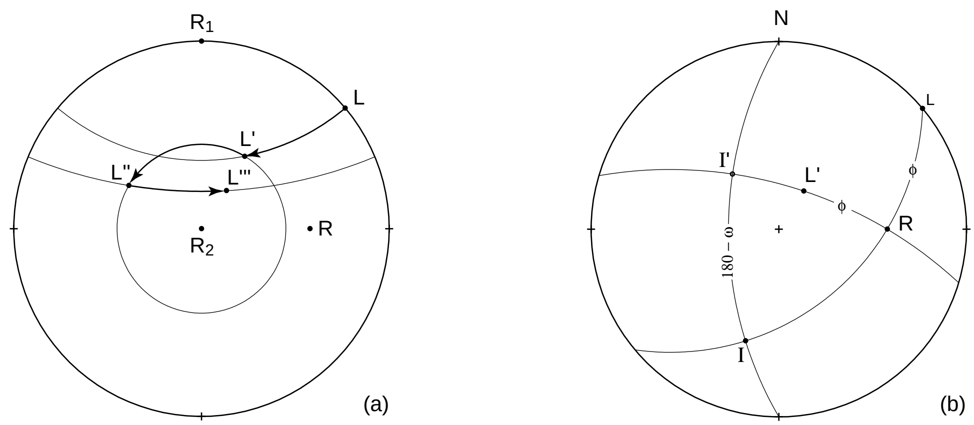

Figure 6.7: Inclined axis: (a) sequential rotations; (b) direct rotation.

Construction I

1. Plot points R and L (Fig. 6.7a).

2. To bring R to the center of the net we need to rotate about an auxiliary horizontal axis whose trend is perpendicular to the trend of R, that is, due north. Mark this point R_1.

3. As R moves $\omega = (90° - 30°) = 60°$ to the center of the net, L moves by the same amount and sense along its small circles to L'.

4. Performing the $\omega = 90°$ anticlockwise rotation about the now vertical axis R_2, point L' moves to L'' along a small circle concentric with the center of the net.

5. Reverse the rotation about R_1 of Step 3 to return R to its original orientation and with it L'' to L'''.

Answer

- The attitude of the line after this sequence of rotations is $L'''(63/033)$.

In serial constructions such as this, the potential errors increase with the number of steps, so this is not the preferred method. However, it is important because it forms the basis of methods which are treated in the next chapter.

The second method involves the direct rotation about the inclined axis using an auxiliary construction (Turner & Weiss, 1963, p. 69). An important advantage is that by reducing the number of steps the plotting errors are also reduced. Imagine the turntable with its axis pointing in the direction of inclined axis R. The plane of the turntable is now inclined and its great circle representation is easily drawn with R as its pole.

Construction II

1. As before plot R and L. Then trace in the great circle normal to R (Fig. 6.7b).
2. Revolve the overlay so that R and L lie on the same great circle on the net. Trace in this arc to intersect the first great circle whose pole is R at I.
3. The angle between L and R along this arc is $\phi = 48°$.
4. As I rotates anticlockwise about R it moves first to the primitive and then its opposite to I'. Therefore count off $\omega = 90°$ from right to left in two increments. Alternatively, count $180° - 90° = 90°$ back from I to locate I'.
5. Revolve overlay so that I' and R lie on the same great circle and count off $\phi = 48°$ from R to locate L'.

Answer

- The attitude of the line after this single rotation is $L'(68/033)$, which is the same as before.

That the rotation of Fig. 6.7b about a single inclined axis produces the same results as the sequence of rotations of Fig. 6.7a illustrates an important fact. By a theorem due to the famous Swiss mathematician Leonard Euler (1707–1783) any sequence of rigid body rotations about a series of axes can always be described by single rotation by a single angle about a single axis.

6.5 ROTATIONAL PROBLEMS

These several rotational techniques solve a class of *forward problems*. In each case, we started with a known initial state, applied a specified rotation, tracked the lines and poles along small circle paths to arrive at the final state. In effect, these model the rotations as they occur in nature.

In contrast, the geologist is faced with quite a different problem. In the field, we observe the orientation of planes and lines which have been rotated in the geologic past. From measurements of such features we wish to determine the rotations which are responsible for these changes in orientation and thus to recover the initial state. These are examples of a class of *inverse problems*. Generally, these are commonly much more difficult to solve.

In particular, the fact that the rigid rotation of a body, no matter how complex, can always be described by a single Euler axis and Euler angle leaves us, in general, with only a description of the angular relation between the initial and final states. There are many motions could have produced a particular difference in orientation from a simple rotation about a single axis to the progressive rotation about a constantly shifting axis. There is no way to distinguish between these on the basis of the measurement of the orientation of line and planes alone. We take up some of these questions again in §7.5–7.6.

6.6 TILTING PROBLEMS

A typical problem involves the restoration of tilted beds and sedimentary lineations which they contain to their pre-tilt orientation. For example, a solution of this problem would aid in the paleogeographic reconstruction of current directions in some past geologic time. We can easily restore the plane to horizontality because it involves the rotation about a horizontal axis, but what about a possible rotation about a vertical axis? We could determine this rotation if we knew the pre-tilt trend of a line, but this is the very question that we are trying to answer.

Knowing only the final state, we usually do not have enough information to recover multiple rotations and the problem is therefore not solvable. What to do? As the construction using sequential rotations indicates, the horizontal component of rotation is parallel to the strike of the tilted beds. As a partial solution we therefore choose R in this direction and then proceed with the restoration on this basis. This is the *conventional tilt correction* (MacDonald, 1980).

Problem

- A dipping bed N 40 W, 60 W contains a sedimentary lineation which trends N 63 W. Restore the bed to horizontality and estimate the original trend of the lineation. Note that the attitude of this inclined plane and line are identical with the forward results obtained in Fig. 6.5a by a sequence of rotations.

Visualization

- Hold the right hand with palm upward and inclined to the west with fingers pointing toward the northwest over the net; also hold a pencil on the palm in the direction of the line. Now rotate the hand through an angle of 60° into a horizontal position and observe the final position of the line.

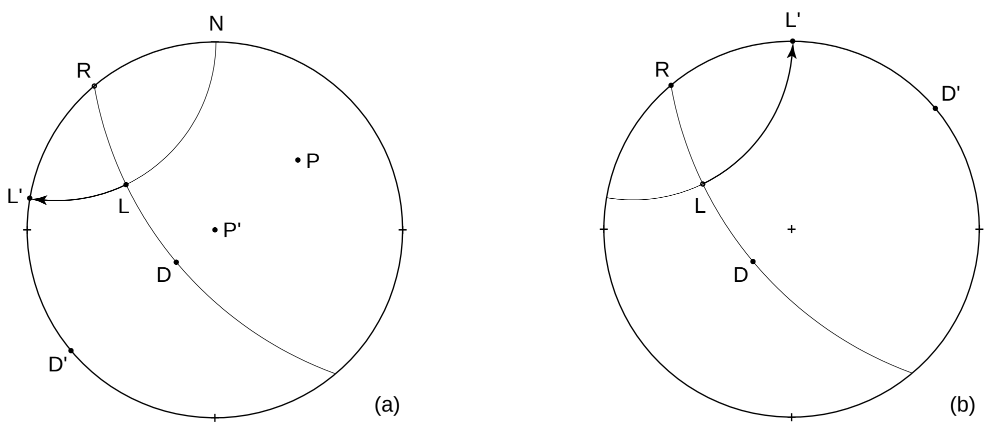

Figure 6.8: Restoration of a plane: (a) upright; (b) overturned.

Construction

1. Trace in the great circle representing the inclined plane and on it locate line $L(34/297)$ (Fig. 6.8a).
2. Plot point $P(30/050)$ representing the pole of the plane.
3. Label the strike direction $R(00/340)$ and turn it to north.
4. As pole P moves 60° to the center of the net (and the bed to horizontality) line L moves along its small circle to L' on the primitive.

Answer

- The restored orientation of the line is $L'(00/280)$.

As we have seen, the rotation of planes generally requires the use of poles. However, in cases such as this where the strike direction is taken as a rotation axis, the line of true dip remains fixed as the line of steepest inclination as if it were a physical line. In this case it is then simpler to rotate this line $D(60/230)$ directly to the primitive, rather than plot and use the pole.[1] We will use this method in the next example.

Clearly, if the tilt correction is not made and the measured trend of the line on the inclined plane is used an error will result. In this example, the difference between the measured trend t and the restored trend t' is $\Delta t = t - t' = 17°$. However, if the angle of dip is small and the line is close to the dip direction this trend error Δt may be negligible (Ten Haaf, 1959, p. 72; Ramsay, 1961). If this approximation is seriously considered it should always be tested with a plot on the stereonet.

Note carefully that this restoration using the conventional tilt correction is not the same as the starting state illustrated in Fig. 6.5. We have not recovered the trend of the line because we have not taken into account the component of rotation about a vertical axis.

Without additional information any such rotation about a vertical axis remains unknown. One way of obtaining such information is to compare the results of the restoration with undisturbed beds nearby. Ten Haaf (1959, p. 78; see also Potter & Pettijohn, 1977, p. 371) used this technique to demonstrate that kilometer-scale coherent slabs in the Appenines of northern Italy rotated about vertical axes through large angles during gravitational sliding. Another approach is to use paleomagnetic vectors to assist in identifying the axis of rotation which restores the tilted beds to their actual initial orientation (Tauxe & Watson, 1994; Weinberger, et al., 1995).

There are certainly situations where beds have been tilted about axes which were horizontal or nearly so and this conventional approach will then produce acceptable results. In the face of the general un certainties, however, it is prudent to remain cautious. All the remaining problems in this chapter involve these same uncertainties.

There is an additional special case. If the plane returned to horizontality was overturned, then the resulting orientation of the associated linear structure obtained using this method will be incorrect. An alternative construction must be used.

Problem

- An overturned sedimentary bed N 40 W, 60 W contains a sedimentary lineation trending N 63 W. Restore the bed to horizontality and determine the original trend of the lineation using the conventional tilt correction.

Visualization

- Hold the left hand, palm downward with a pencil in the proper orientation over the net. Now rotate the hand through an angle of 120° into a horizontal position with the palm upward and observe the position of the line.

Construction

1. Trace in the great circle representing the inclined plane and on it locate the line of true dip $D(60/230)$ and the line $L(34/297)$ (Fig. 6.8b).

[1] That this is not true for rotations about other axes see Fig. 6.4 where point L_2 is located on the line of true dip on its plane but the rotated point L_2' is *not* on the line of true dip of the rotated plane.

2. Label the strike direction $R(00/340)$ and turn this mark to north.
3. As D moves 120°, first to center of the net and then to the primitive (and the bed to horizontal), L moves along its small circle to L' in the same sense and angle, also to the primitive.

Answer

- The attitude of the restored line is $L'(00/000)$, that is, horizontal and due north. This is quite different result from the restoration in upright case.

6.7 TWO TILTS

A closely related situation involves the restoration of a structural plane that has been tilted twice, called the *problem of two tilts*. The goal is to determine the attitude the plane after the first but before the second tilt.

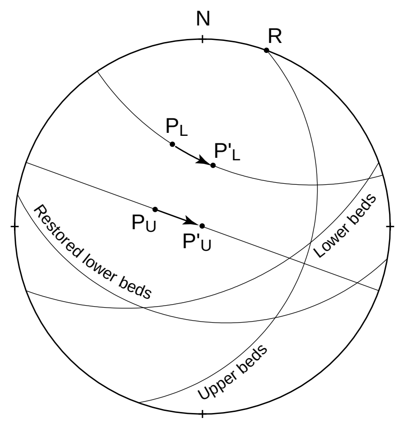

Figure 6.9: Problem of two tilts.

Problem

- The attitude of beds above an angular unconformity is N 20 E, 30 E and the attitude of the beds below the unconformity is N 70 E, 50 S. What was the attitude of the lower beds before the tilt of the upper beds?

Method

- Because this correction of the tilt of the lower beds involves rotation about an axis which is oblique to the strike direction we can not used the line of true dip. The representation of the planes by their poles is required. Plotting the tilted planes by great circles is not necessary for a solution but they are helpful in the visualizing the procedure and result.

Construction

1. Plot the poles of the once tilted upper beds $P_U(60/290)$ and the twice tilted lower beds $P_L(40/340)$ (Fig. 6.9).

2. Mark the strike direction of the upper beds as the rotation axis $R(00/020)$ and turn it to north.

3. In restoring the upper beds to horizontality, pole P_U moves 30° inward to the center of the net and pole P_L moves 30° in the same direction along its small circle to P'_L.

Answer

- The restored pole $P'_L(53/010)$ of the lower beds corresponds an attitude of N 80 W, 37 S.

6.8 FOLDING PROBLEMS

These same techniques can also be used to restore the attitude of folded beds. We treat the details of the geometry of folds in Chapter 13. Here it is sufficient to treat folds as to two planes whose line of intersection represents the fold axis. In such applications there is an important caveat. If the folding is accompanied by distortion of the bedding planes the angular relationships change and this requires a more involved treatment (see §12.9; also Ramsay, 1961). The following treatment assumes that such distortions are absent.

If the folds are horizontal the conventional tilt correction suffices to return the beds to horizontality. If the folds plunge, the tilted beds can be considered to have two rotational axes: one of them is the fold axis and the other is a horizontal axis perpendicular to the trend of the fold axis (Ramsay, 1961). Reversing the rotations on both these then unrolls the folded beds to their original orientation. Using the previous approach, a sequence of rotations is used. First, the beds are unrolled about the plunging fold axis and then about the resulting strike direction to bring the beds back to horizontal.

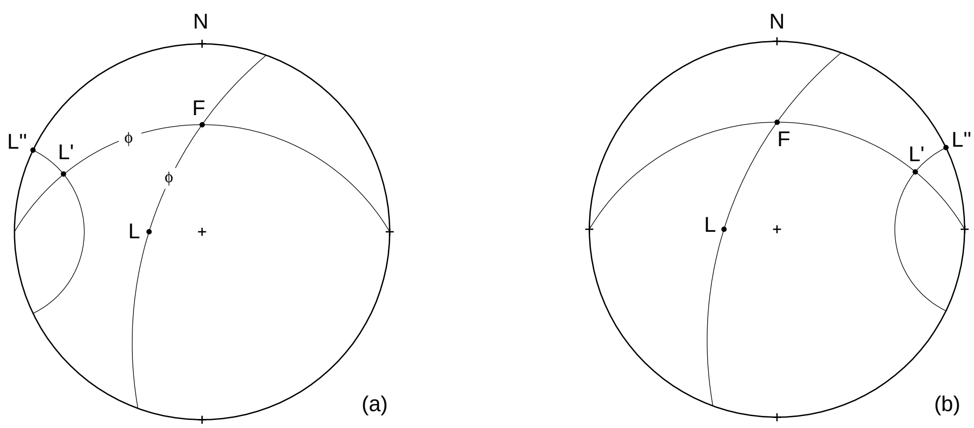

Figure 6.10: Restoring folded beds: (a) upright limb; (b) overturned limb.

Problem

- The fold axis plunges 31° due north. On the west limb of an anticline, inclined beds whose attitude is N 20 E, 60 W contain sole markings which trend due west. Determine the prefolding orientation of this sedimentary lineation.

Visualization

- With the left hand represent the plane on the west limb with the index finger in the direction of the fold axis. Similarly, with the right hand represent a similarly oriented plane on the east limb. Now rotate both planes about your index fingers to bring the two planes into parallelism. Now perform the tilt correction to bring this plane into horizontality.

Construction

1. Plot the fold axis $F(30/000)$ and draw in the great circle representing the inclined plane and locate the east-trend line L on its trace. Note that this great circle must pass through F (Fig. 6.10a).
2. Read off the angle ϕ between L and F.
3. Unrolling the beds about the plunging fold axis results in a plane dipping 30° due north. The angle between F and L $\phi = 55°$ remains constant. Thus after unfolding L' can be located at the same angle along the great circle representing this north-dipping plane.
4. The tilt correction then brings the plane to horizontal and the line to L'' on the primitive.

Answer

- The restored orientation of the sedimentary lineation is $L''(00/056)$.

If the beds are overturned this simple restoration will be in error and an adjustment must be made. With the same visualization as before, now rotate your hand about the fold axis so that the palm is downward. The angle ϕ remains the same, but now L' is on the opposite side of F. The tilt correction then gives L''' with a trend of N 56 E (Fig. 6.10b).

6.9 SMALL CIRCLES

Throughout this chapter we have made use of small circles on the stereonet. It is a fundamental property of the stereographic projection that circles on the sphere project as circles (see §5.1). Here, we show how to construct a small circle about any inclined axis. We also prove that they are indeed circles.

As we have seen, a small circle is the intersection of the sphere and a right-circular cone. A vertical diametral plane of the sphere containing the inclined axis OP displays a section MON of this cone (Fig. 6.11a). The cone axis makes an angle θ with the vertical and its vertex angle is 2ϕ. Line MN is the trace of the circular section of this cone.

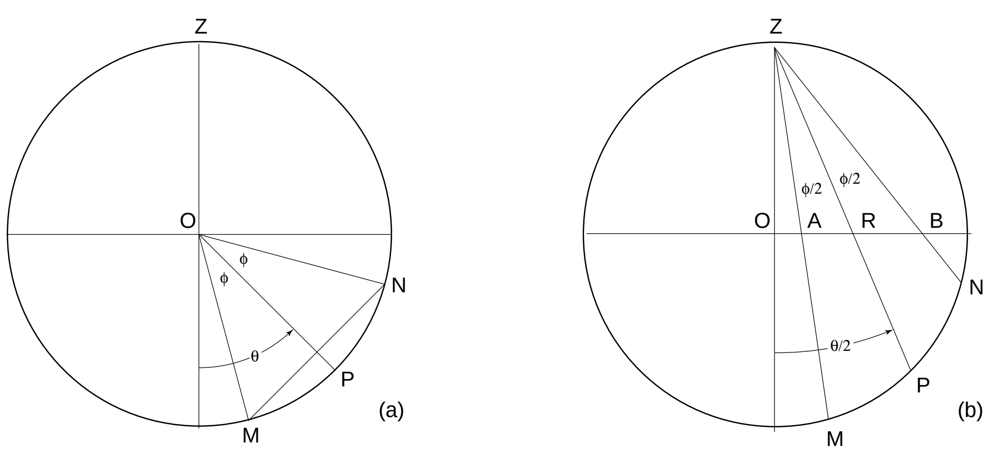

Figure 6.11: Small circle: (a) on sphere; (b) in projection.

We project the small circle on the sphere to the horizontal projection plane using the zenith point Z. The center of the circle P projects to R, the lowest point M and highest point N on the circle project to points A and B (Fig. 6.11b). With these two points any small circle may be drawn on the stereonet. There are two important cases. The circle may be wholly within the lower hemisphere or it may be partially in the upper hemisphere. We start with the simpler case when the cone is entirely within the lower hemisphere.

Problem

- Construct the small circle whose angular radius $\phi = 35°$ about inclined axis $R(45/140)$.

Construction

1. Plot the axis $R(45/160)$. On a radius of the net through R plot points A and B at $\phi = 35°$ measured from R.
2. Bisect the linear distance AB to locate the center C and complete the circle with radius $AC = BC$ using a compass. Note that C does not coincide with R (Fig. 6.12).

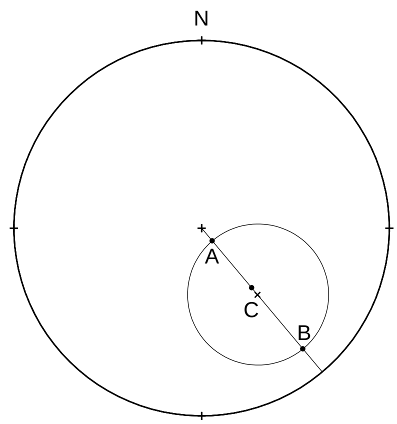

Figure 6.12: Construction of a small circle.

If the small circle overlaps the primitive, that is, if it extends partially into the upper hemisphere it is then necessary to construct the arc of its *opposite*. This requires additional steps because there is no direct way of plotting points outside the primitive.

On the vertical diametral section of the sphere, the trace of the cone is MON and its opposite is $M'ON'$ (Fig. 6.13). Points M and N are projected using Z to points A and B on the projection plane in the usual way, A inside and B outside the primitive. In the same way the opposite points M' and N' are projected to A' outside and B' inside the primitive. Note that $\angle NZN' = \angle MZM' = 90°$.

Just as before, segments AB and $A'B'$ are bisected to locate centers C and C' and the two circles are completed with a compass. Although it is in two parts, the small circle is now complete in the lower hemisphere (and also in the upper hemisphere).

Expressions for the location of the center of the small circle and its radius can also be obtained. For a sphere of unit radius, and in the notation of Fig. 6.11b,

$$OA = \tan \tfrac{1}{2}(\theta - \phi) \quad \text{and} \quad OB = \tan \tfrac{1}{2}(\theta + \phi),$$

where θ is the supplement of the plunge of the cone axis and ϕ is the semi-vertex angle of the cone. With these the distance from O to the geometrical center of the small circle on the projection plane is then

$$c = \tfrac{1}{2}(OB + OA) = \tfrac{1}{2}\left[\tan \tfrac{1}{2}(\theta + \phi) + \tan \tfrac{1}{2}(\theta - \phi)\right]$$

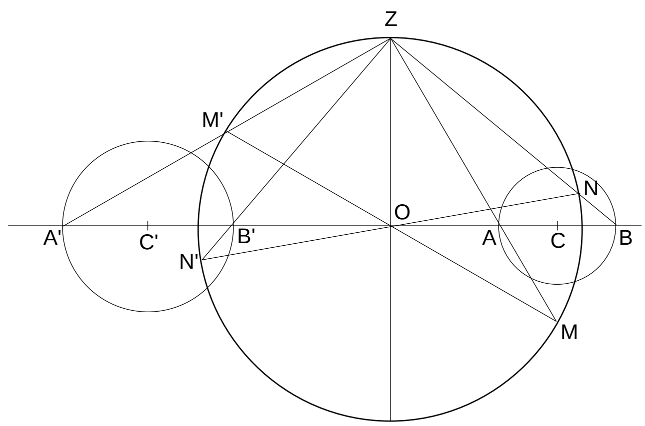

Figure 6.13: Small circle and its opposite.

and its radius is

$$r = \tfrac{1}{2}(OB - OA) = \tfrac{1}{2}\left[\tan\tfrac{1}{2}(\theta+\phi) - \tan\tfrac{1}{2}(\theta-\phi)\right].$$

Substituting the identities

$$\tan\tfrac{1}{2}(\theta+\phi) = \frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} \quad \text{and} \quad \tan\tfrac{1}{2}(\theta-\phi) = \frac{\sin\theta-\sin\phi}{\cos\theta+\cos\phi}$$

and rearranging, these two expressions become

$$c = \frac{\sin\theta}{\cos\theta+\cos\phi} \quad \text{and} \quad r = \frac{\sin\phi}{\cos\theta+\cos\phi}.$$

These equations can also be used to locate earthquake epicenters (Garland, 1979, p. 54). Because $\theta = (90^\circ - p)$ a more convenient form for our purposes is

$$c = \frac{\cos p}{\sin p+\cos\phi} \quad \text{and} \quad r = \frac{\sin\phi}{\sin p+\cos\phi}. \tag{6.1}$$

With these the location and size of a circle which is mostly in the lower hemisphere can be easily determined for a stereogram of any size. Just multiply the values of both c and r by the desired radius of the primitive.

These two parameters can also be used to calculate the location and size of the opposite small circle by using $-p$ (indicating an upward inclination of the cone axis) in Eqs. 6.1, or by using

$$c = \frac{\cos p}{-\sin p+\cos\phi} \quad \text{and} \quad r = \frac{\sin\phi}{-\sin p+\cos\phi}. \tag{6.2}$$

Both the graphical and analytical methods illustrate two aspects of opposite small circle which have some practical importance.

1. As the opposite point M' approaches the projection point Z both c and r become very large and drawing the opposite arc is difficult or impossible.

2. Opposite small circles have two basic configurations:

   (a) If $(p+\phi) < 90^\circ$ its arc is convex toward the center of the net (Fig. 6.14a)

   (b) If $(p+\phi) > 90^\circ$ its arc is concave toward the center (Fig. 6.14c).

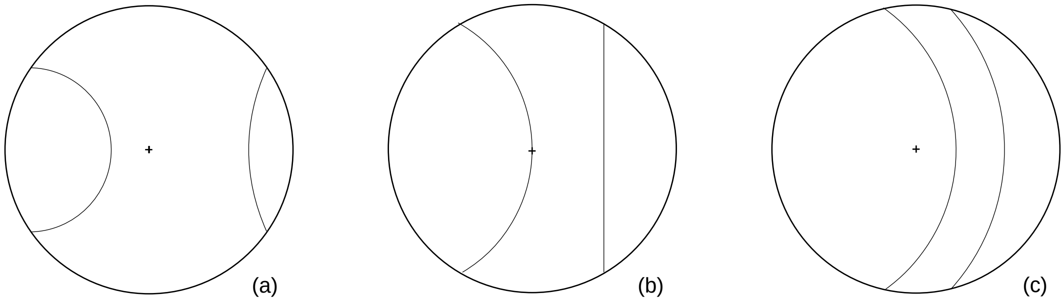

Figure 6.14: Types of small circles.

The boundary case occurs when the low point on the cone coincides with the center of the net, that is, when $(p + \phi) = 90°$ (Fig. 6.14b). In the graphical construction of its opposite A', the projector from Z is parallel to the projection plane and both c and r are infinite. In Eqs. 6.2 this state is indicated when the denominator $(-\sin p + \cos \phi) = 0$. The representation of the opposite of such a circle is particularly easy to construct — it is a straight line.

We now demonstrate that circles on the sphere do, in fact, project as circles following Phillips (1963, p. 24–25). As we have seen, any small circle is the intersection of a sphere and a right-circular cone with vertex at the center of the sphere (Fig. 6.11a). The axis of this cone OP makes angle θ with the vertical and $\angle MON = 2\phi$. The small circle on the sphere has a diameter of MN and the point P is at its center.

The projection of points P, M and N on the sphere to the projection plane uses the zenith point Z. The resulting three points A, B and R define a second cone whose axis ZP makes an angle $\frac{1}{2}\theta$ with the vertical and $\angle MZN = \phi$ (Fig. 6.11b).

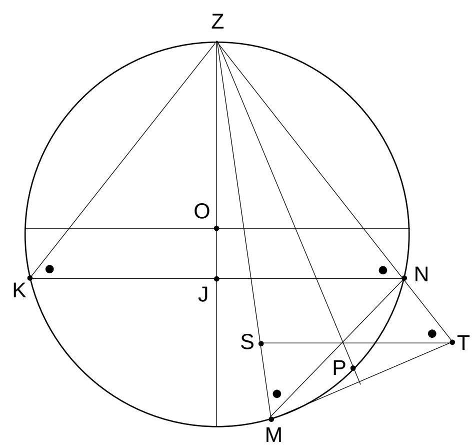

Figure 6.15: A small circle and its projection.

Proof

1. On the vertical plane containing the cone axis, chord KN is drawn parallel to the projection plane, hence also perpendicular to OZ. Right triangles ZKJ and ZNJ are congruent and so $\angle ZKJ = \angle ZNJ$ (Fig. 6.15, where black dots mark equal angles).

2. Because they subtend these equal angles, the lengths of arcs ZK and ZN are equal. Then the inscribed angles which subtend these equal arcs are also equal, so $\angle ZMN = \angle ZNJ$.

3. Line MN, which is oblique to axis ZP, is the trace of a circular section of cone MZN. Therefore the right section of this cone MT is an ellipse.

4. By construction TS make the same angle with axis ZP as the circular section MN. Therefore TS is a conjugate circular section.

5. Therefore $\angle ZTS = \angle ZMN$, and also $\angle ZNJ = \angle ZTS$. Lines TS and NJ are then parallel and also parallel to the projection plane.

6. Parallel sections of a cone are similar. Therefore the section in the projection plane is also a circle and the proof is complete.

6.10 EXERCISES

1. A horizontal plane contains a line whose trend is N 48 E.
   (a) Rotate the plane and line about a vertical axis 50° anticlockwise.
   (b) From the same starting position, rotate the plane and line about a north-trending horizontal axis 60° clockwise.

2. Sequence of rotations.
   (a) Rotate the same horizontal plane and line first about a vertical axis 50° anticlockwise and then about a north-trending horizontal axis 60° clockwise.
   (b) Rotate the horizontal plane and line first about a north-trending horizontal axis 60° clockwise and then about a vertical axis 50° anticlockwise.

3. Rotate the same plane and line about an axis whose plunge and trend is 30/200 40° clockwise.

4. The beds below an angular unconformity have an attitude of N 30 W, 40 W. The strata above the unconformity have an attitude of N 20 E, 30 E. What was the attitude of the lower beds before the tilting of the younger bed occurred?

5. An anticlinal fold axis plunges 24/040. On the east limb where the beds have an attitude of N 5 W, 32 E, the crest line of current ripple marks pitches 70 N in the plane of the bedding. What was the pretilt orientation of these marks? Compare your result with the assumption that the tilted orientation of the lineation adequately represents the original direction.

6. An anticline plunges 50/025. The eastern limb is overturned, and at one point the attitude is N 45 E, 50 W. At this same locality a sedimentary lineation plunges due west. What was the orientation of the lineation before folding?

7. Rotate the plane whose attitude is N 10 E, 30 E, fifty degrees anticlockwise as viewed down the plunge of an axis whose attitude is 30/340 in two ways: (1) as a series of steps involving rotation of the axis to the primitive, rotating the line about the now horizontal axis, and then returning the axis to its original orientation; (2) as a single rotation about the inclined axis.

8. MORE