Chapter 7

VECTORS

7.1 INTRODUCTION

Vectors play a prominent role in many geometrical and physical applications. We have seen several simple examples in Chapters 1 and 3 where the attitudes of planes and lines were represented and manipulated using two-dimensional vectors. We now extend the treatment to three-dimensions, and to several additional applications.

As we have seen, the stereonet is a useful way of displaying and manipulating structural lines and planes easily and directly in a three-dimensional setting. For the same reasons, we can use the stereonet to introduce an analytical approach involving vectors which is a powerful method for solving these same types of problems (see also Sprenke, 1992).

Because the orientation of planes are defined by their poles, we can represent all structural elements by lines. There are two types of such lines.

1. *Axes* have orientation but no sense. Lineations in metamorphic rocks, lines of intersection and poles of fracture planes are examples.

2. *Vectors* have both orientation and sense. Examples include some linear sedimentary structures and paleomagnetic directions.

Some structural lines may be treated in either way. For purely geometric purposes the pole of sedimentary bedding is commonly treated as an axis, but for other purposes the pole in the direction of younging has sense and therefore is a true vector.

In many applications it is convenient to represent axes by vectors. In fact, we have already done this in Chapters 1 and 3 by choosing to represent lines of true and apparent dip as horizontal vectors which point in the direction of downward inclination. As before, the sense of these vectors is arbitrarily but conveniently chosen to point *downward*, thus we can always plot them on the lower hemisphere. If we encounter an upward pointing axis-as-vector we can immediately convert it to a downward pointing one.

These vectors can then be manipulated by taking advantage of the well-established vector formalism encountered in introductory courses in calculus and physics. Not only is this a particularly

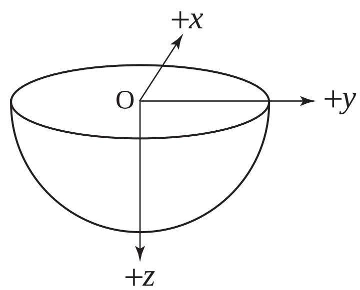

Figure 7.1: Lower hemisphere and the coordinate axes.

powerful way of solving a number of structural problems but it also lays the groundwork for more advanced applications (Goodman, 1976, p. 217f; Priest, 1985; Wallbrecher, 1986[1]

We need a coordinate system. For problems involving conditions within the earth, it is nearly universal to measure depth along a downward pointing z axis. Accordingly, we define a right-handed set of axes with $+x =$ north, $+y =$ east and $+z =$ down (Fig. 7.1).[2] The equation of the unit sphere is then

$$x^2 + y^2 + z^2 = 1, \tag{7.1}$$

and the directions of vectors can be represented by points on the surface of this sphere.

In many applications it is necessary to use vector components. Following common practice we take unit base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to be parallel to the coordinate directions $+x$, $+y$ and $+z$, respectively. Then any vector $\mathbf{V}$ is the sum of its components in each of these directions.[3] We write this sum as

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}. \tag{7.2}$$

Following common practice, we represent such a vector by its three scalar components (V_x, V_y, V_z). The length or magnitude of vector $\mathbf{V}$ is, from a three-dimensional version of the Pythagorean theorem,

$$|\mathbf{V}| = V = \sqrt{V_x^2 + V_y^2 + V_z^2}. \tag{7.3}$$

When used to represent the orientation of lines and poles we are interested in the directions not magnitudes of the associated vectors. It is then convenient to use only vectors of unit magnitudes. To find the unit vector with the same direction as a general vector, we *normalize* its components by dividing each by the magnitude. The three scalar components of this unit vector are called *direction cosines* and they are commonly given the symbols l, m and n, where

$$l = V_x/V, \quad m = V_y/V, \quad n = V_z/V. \tag{7.4}$$

[1] Wallbrecher's book contains the listing of a number of useful programs for structural geology. These are now available as the package Fabric7 at `www.geolsoft.com`.

[2] In contrast, for problems involving surface or near surface features a geographical coordinate system with $+x =$ east, $+y =$ north and $+z =$ up is generally used (see §7.8).

[3] We use the symbol $\mathbf{V}$ to represent a generic vector. It should not be confused with the commonly used symbol for *volume*.

With these, Eq. 7.3 reduces to the useful identity

$$l^2 + m^2 + n^2 = 1. \tag{7.5}$$

With any two direction cosines this relationship yields the magnitude but not the sign of the third.

For use in plotting vectors as points on the stereonet *direction angles* are more useful and these are defined as

$$\alpha = \arccos l, \quad \beta = \arccos m, \quad \gamma = \arccos n. \tag{7.6}$$

From the point plotted using its plunge and trend we can measure these three direction angles on a stereogram.

Problem

- Find the direction angles of the vector $\mathbf{V}(30/300)$.

Solution

1. Using its plunge and trend plot the point representing $\mathbf{V}$ in the usual way (Fig. 7.2a).
2. Measure α from $+x$, β from $+y$, and γ from $+z$ to V along great circular arcs.

Answer

- The direction angles are $\alpha = 64°$, $\beta = 139°$ and $\gamma = 60°$.

These angles can be checked by using Eq. 7.5, but note that this identity will rarely be exactly satisfied because of inevitable errors in plotting the points and reading the angles. In the previous example the sum is 1.01176, and this is about as close to confirmation as one can get when reading angles from the net to the nearest degree.

A closely related problem is to plot a vector on a stereogram given its direction angles.

Problem

- Plot the point representing $\mathbf{V}$ using its direction cosines $l = 0.43301$, $m = -0.75000$, $n = 0.50000$.

Solution

1. With Eqs. 7.6, the three direction angles are $\alpha = 64.3°$, $\beta = 138.6°$, $\gamma = 60.0°$.
2. About the point on the primitive representing the $+x$ axis trace in the small circle with angular radius $\alpha = 64.3°$ (Fig. 7.2b).

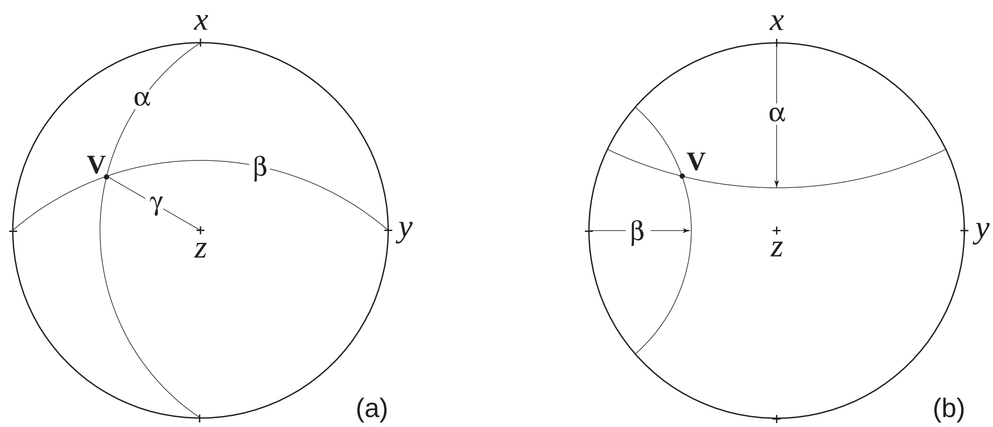

Figure 7.2: Direction angles: (a) reading angles; (b) plotting angles.

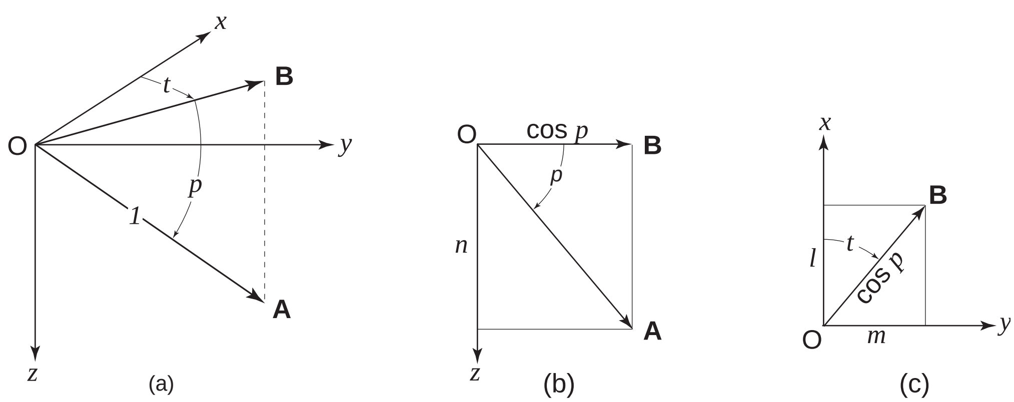

Figure 7.3: Direction cosines from plunge and trend.

3. About the point representing the $+y$ axis trace in the small circle with angular radius $\beta = 138.6°$. Alternatively, this small circle can be located $180° - 138.6° = 41.4°$ from $-y$. Even simpler, just change the sign of m and then $\arccos(+0.75000) = 41.4°$.

4. The small circle for $\gamma = 60.0°$ about $+z$ can be added to the diagram with a compass but it is usually unnecessary. It is, however, a good idea to check that angle γ between $+z$ and the intersection of the other two small circles is correct.

As with structural lines generally, we may also express the orientation of vectors by their plunge p and trend t. Note that in our adopted coordinate system *positive trend angles* are measured clockwise from $+x =$ north and *positive plunge angles* are measured downward from the horizontal xy plane, as is standard. On the stereonet these angles are closely related to spherical coordinates: $\theta = t$, $\phi = 90° - p$ and $r = 1$.

From plunge and trend we can also compute the direction cosines (Fig. 7.3a). The horizontal component of the inclined unit vector $\mathbf{A}$ with length $OA = 1$ is vector $\mathbf{B}$ where $B = \cos p$. From

Figs. 7.3b and 7.3c, the direction cosines of $\mathbf{A}$ are

$$l = \cos p \cos t, \quad m = \cos p \sin t, \quad n = \sin p. \tag{7.7}$$

These may also be converted back to plunge and trend with

$$p = \arcsin n, \quad t = \arctan m/l. \tag{7.8}$$

The *arctan* function on hand-held calculators and in most programming languages returns angles in the range $-90° < t < 90°$, that is, only trend angles in the NE and NW quadrants are reported. If $l < 0$ then the actual angle is in range $90° < t < 270°$ and it is necessary to add or subtract $180°$ to get the correct trend. In most programming languages and spreadsheet programs, the alternative function `atan2(m,l)` gives the trend without need for this correction.

7.2 SUM OF VECTORS

There are many physical and geometric situations where two or more vectors must be combined by addition, as we will see later. The equation expressing the addition of two vectors to give a third is

$$\mathbf{A} + \mathbf{B} = \mathbf{C}. \tag{7.9}$$

Given vectors $\mathbf{A}$ and $\mathbf{B}$ we can determine their sum $\mathbf{C}$ either geometrically or analytically. The geometrical method uses the *parallelogram rule*: Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$. Then draw the vector from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$; this is $\mathbf{C}$ (Fig. 7.4a). Note that $\mathbf{B}+\mathbf{A}$ gives the same result — vector addition is commutative. Note too that $\mathbf{C}$ is the diagonal of the parallelogram with sides parallel to $\mathbf{A}$ and $\mathbf{B}$, hence the name of the rule.

The difference of two vectors can also be found. The solution of Eq. 7.9 for $\mathbf{A}$ can be written in two ways

$$\mathbf{A} = \mathbf{C} - \mathbf{B} \quad \text{or} \quad \mathbf{A} = \mathbf{C} + (-\mathbf{B}).$$

The vector $-\mathbf{B}$ has the same length as $\mathbf{B}$ but points in the opposite direction. The graphical solution proceeds just as before (Fig. 7.4b).

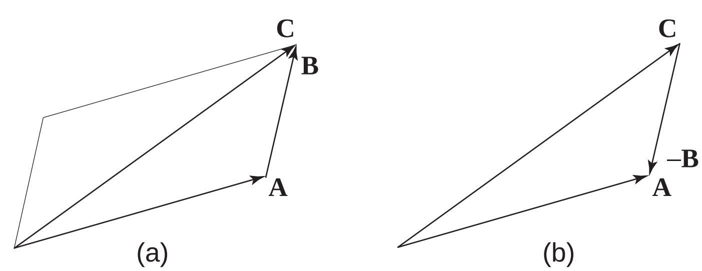

Figure 7.4: Parallelogram rule: (a) addition; (b) subtraction.

The analytical method involves representing vectors as matrices. This enumerates the components and at the same time emphasizes that they represent a single entity. Even more important is that such matrices can be manipulated directly using matrix algebra.[4] In a simple example we represent them as *column matrices*. Then we form the sum of two vectors by adding components (Fig. 7.5). This is expressed by the matrix equation

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix}. \tag{7.10}$$

The extension to three dimensions is straight forward.

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{bmatrix}. \tag{7.11}$$

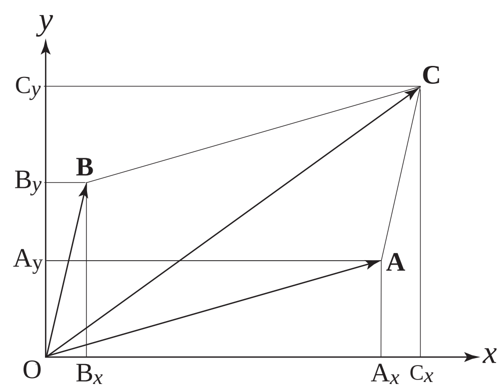

Figure 7.5: Vector addition using components.

A useful application involves finding the vector which bisects the angle between two given vectors. If $A = B$ then $\mathbf{C}$ divides the parallelogram whose sides are $\mathbf{A}$ and $\mathbf{B}$ into two congruent triangles. Therefore the angles between $\mathbf{A}$ and $\mathbf{C}$ and between $\mathbf{B}$ and $\mathbf{C}$ are equal (Fig. 7.6a).

Problem

- Bisect the angle between vectors $\mathbf{A}(30/310)$ and $\mathbf{B}(60/030)$ (Fig. 7.6b).

Solution

1. From plunge and trend of each vector, the direction cosines are $\mathbf{A}(0.55667, -0.66341, 0.50000)$ and $\mathbf{B}(0.43301, 0.25000, 0.86603)$.

[4] Good geologically-oriented introductions to matrix algebra are given by Ferguson (1994) and Davis (2002, p. 123–158).

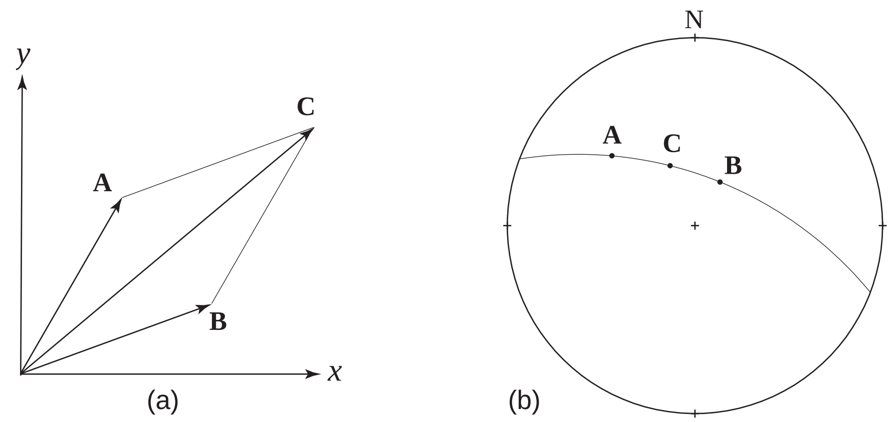

Figure 7.6: Bisector of two vectors: (a) two dimensions; (b) three dimensions.

2. From Eq. 7.11 we then have

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} 0.98968 \\ -0.41341 \\ 1.36603 \end{bmatrix}. \tag{7.12}$$

3. Normalizing these components of $\mathbf{C}$ we obtain the direction cosines $\mathbf{C}(0.56984, -0.23803, 0.78653)$.

Answer

- The plunge and trend of the bisector is $\mathbf{C}(52/337)$. If vectors $\mathbf{A}$ and $\mathbf{B}$ represent the poles of planes, then vector $\mathbf{C}$ bisects the angle between the two planes.

7.3 PRODUCTS OF VECTORS

Important relationships between two vectors can be found by forming the *scalar* or *dot product* and the *vector* or *cross product.*

DOT PRODUCT

The first and simpler product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined in terms of their magnitudes A and B and the angle ϕ between them as

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\phi, \tag{7.13}$$

where $0 \le \phi \le 90°$. This has a useful geometrical interpretation: $B\cos\phi$ is the projection of $\mathbf{B}$ onto $\mathbf{A}$ and $A\cos\phi$ is the projection of $\mathbf{A}$ onto $\mathbf{B}$. The dot product can also be expressed in component form as

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{7.14}$$

From these two versions it should be apparent that the order in which the vectors are taken makes no difference, that is, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. The dot product, like the sum, is commutative.

As before, it is convenient to write such expressions as matrix equations. Here $\mathbf{A}$ is represented by a *row matrix* and $\mathbf{B}$ by a *column matrix*. Thus

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_x B_x + A_y B_y + A_z B_z. \tag{7.15}$$

In this easily remembered form the three resulting scalar quantity is obtained by summing the products of the corresponding elements of the row and the column matrices. This is an example of *row times column multiplication* (Boas, 1983, p. 115–116), and we will use it repeatedly.

If both vectors have unit magnitudes then Eqs. 7.13 and 7.14 combine to give the useful formula for finding the angle between them

$$\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 = \cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2, \tag{7.16}$$

where (l_1, m_1, n_1) and (l_2, m_2, n_2) are the two sets of direction cosines. With this formula the angle between any two directions represented by unit vectors may be easily found. If $\theta = 90°$, that is the two vectors are mutually perpendicular, this equation reduces to

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \tag{7.17}$$

and this can be used as a test for orthogonality.

The dot product can also be used if the two unit vectors lie in one of the coordinate planes. For example, in the xy plane, the direction angles measured from $+z$ are $\gamma_1 = \gamma_2 = 90°$ and therefore $n_1 = n_2 = 0$. Then Eq. 7.16 reduces to

$$\cos\phi = l_1 l_2 + m_1 m_2, \tag{7.18}$$

and similar results can be obtained for vectors in the yz and zx coordinate planes.

Problem

- What is the angle between the pole vectors $\mathbf{P}_1(30/310)$ and $\mathbf{P}_2(60/030)$?

Solution

1. From each plunge and trend, using Eqs. 7.7, the direction cosines are

$$\mathbf{P}_1(0.55667, -0.66341, 0.50000) \quad \text{and} \quad \mathbf{P}_2(0.43301, 0.25000, 0.86603)$$

0.51503

2. With Eq. 7.16 $\cos\phi = \cancel{0.59820}$ or $\phi = 59°$. This angle may be acute or obtuse; if acute, as here, it is the dihedral angle between the two planes (Fig. 7.7a).

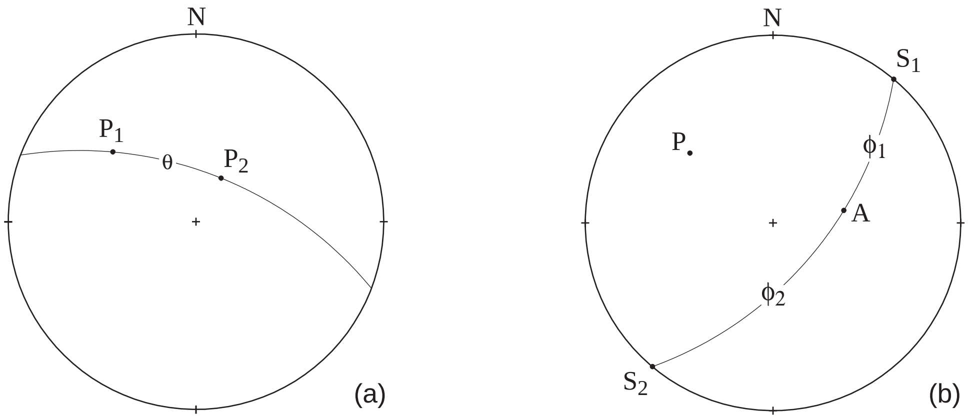

Figure 7.7: Dot product: (a) angle between vectors; (b) pitch of a line.

In exactly the same way the angle between two lines $\mathbf{L}_1$ and $\mathbf{L}_2$ or between a line $\mathbf{L}$ and a pole $\mathbf{P}$ may also be found from the dot product.

This technique can also be used to calculate the pitch of a line in a specified plane. There are several ways of doing this but the one which corresponds most closely with the previous graphical method requires the direction of the strike of the plane — it is perpendicular to the trend of the pole vector or equivalently, perpendicular to the trend of the dip vector. The orientation of this strike vector can be obtained in either of two ways.

1. From the trend t of either the pole or the dip vector, the strike direction is simply $t + 90^\circ$ or $t - 90^\circ$.

2. The direction of the strike may also be determined from the direction cosines (l, m, n) of either the pole or dip vector. From Eqs. 7.8, trend depends on m/l. The strike is then obtained from the negative reciprocal $-(l/m)$. This horizontal strike vector has two forms corresponding to its two equivalent ends

$$\mathbf{S}_1(-m, l, 0) \quad \text{and} \quad \mathbf{S}_2(m, -l, 0). \tag{7.19}$$

Problem

- The orientation of a plane is given by its pole vector $\mathbf{P}(30/310)$. Determine the pitch of the apparent dip vector $\mathbf{A}(48/080)$ (Fig. 7.7b).

Solution

1. The two possible strikes of the plane are $\mathbf{S}_1(00/040)$ and $\mathbf{S}_2(00/220)$. From Eqs. 7.7, the components of the unit strike vectors are $\mathbf{S}_1(0.76604, 0.64279, 0.00000)$ and $\mathbf{S}_2(-0.76604, -0.64279, 0.0000$

2. From its plunge and trend, the components of the apparent dip vector are $\mathbf{A}(0.11619, 0.65897, 0.74314)$.

3. Eq. 7.16 gives $\cos\phi_1 = 0.51258$ or $\phi_1 = 59°$ measured from $\mathbf{S}_1$ and $\cos\phi_2 = -0.52158$ or $\theta_2 = 121°$ measured from $\mathbf{S}_2$. Note that $\phi_1 + \phi_2 = 180°$. By convention the pitch angle is acute.

CROSS PRODUCT

The second way of forming the product of two vectors is written

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}. \tag{7.20}$$

The product vector $\mathbf{C}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and its direction is determined by the *right-hand rule:* if the fingers of the right hand point from $\mathbf{A}$ toward $\mathbf{B}$ through the smaller angle, the thumb points in the direction of $\mathbf{C}$. If the order is reversed, the direction of $\mathbf{C}$ is also reversed, hence the order does make a difference. This condition can be express as $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$. In other words, the cross product is not commutative.

The magnitude of the cross product vector is defined as

$$C = AB\sin\phi, \tag{7.21}$$

where, as before, ϕ is the smaller angle between the two vectors.

In component form, the cross product may be expressed as the easily remembered determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

For computational purposes this 3×3 determinant can be reduced to the sum of three 2×2 determinants by the *method of expansion by cofactors*, which follows a simple rule: for each scalar coefficient cross out in turn the row and column containing the unit base vectors $\mathbf{i}$, $\mathbf{j}$ or $\mathbf{k}$ and in each case form the determinant of the remaining four elements. For example, by crossing out the first row and first column gives the 2×2 determinant composed of the four remaining elements times $\mathbf{i}$ (Fig. 7.8a). The other cofactors are found in similar fashion. The full result is

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \mathbf{k}. \tag{7.22}$$

Note that the way the signs alternate follows a simple pattern: if the sum of the row number and the column number is *even* the sign is positive and if *odd* the sign is negative.

Expanding these three separate 2×2 determinants also follows an easily remembered pattern: form the product of the upper-left and lower-right elements and subtract the product of the upper-right and lower-left elements (Fig. 7.8b). Applying this rule we then obtain

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}. \tag{7.23}$$

Thus

$$C_x = A_yB_z - A_zB_y, \quad C_y = A_zB_x - A_xB_z, \quad C_z = A_xB_y - A_yB_x. \tag{7.24}$$

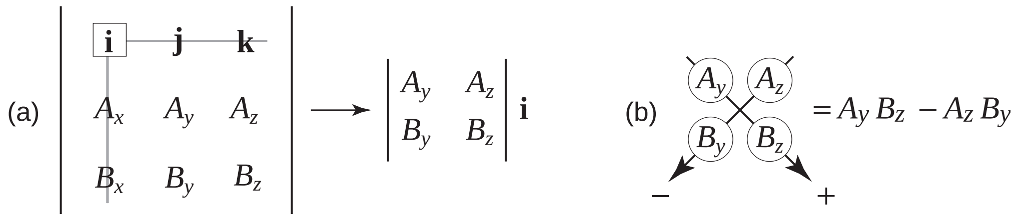

Figure 7.8: Cofactors and determinants.

These expressions apply fully to any set of coordinate axes. As is often the case a special set of axes brings out some important aspects simply and clearly. Thus it is convenient to choose axes so that the plane of the two vectors $\mathbf{A}$ and $\mathbf{B}$ coincides with the xy coordinate plane. We can then see that the cross product has an important geometrical interpretation: in Fig. 7.9a the magnitude of vector C represents the area of the *parallelogram* with sides parallel to $\mathbf{A}$ and $\mathbf{B}$, that is

$$C = Ah = AB\sin\phi.$$

This is identical with the definition of Eq. 7.21. Thus the vector $\mathbf{C}$ represents the orientation of the plane of the parallelogram and its magnitude C represents its area.

It is also of some interest to express this area in terms of the components of the vectors $\mathbf{A}$ and $\mathbf{B}$. Dividing the parallelogram into two parts by a diagonal gives two congruent isosceles triangles which have identical areas (Fig. 7.9b). The area of these identical triangles is found from the sum of a right triangle (Fig. 7.9c) and a trapezoid (Fig. 7.9d) less the area of a second right triangle (Fig. 7.9e). From these three figures

1. Area of the first sub-triangle is equal to half the base times the height $+\frac{1}{2}(B_xB_y)$ (Fig. 7.9c).
2. Area of the trapezoid is equal to the base times the mean height $+\frac{1}{2}(A_x - B_x)(A_y + B_y)$ (Fig. 7.9d).
3. Area of the second sub-triangle has an area of $-\frac{1}{2}(A_xA_y)$ (Fig. 7.9e).

After summing these three expressions, multiplying by 2, expanding and collecting terms, the total area of the parallelogram is then given by

$$B_xB_y + (A_x - B_x)(A_y + B_y) - A_xA_y = A_xB_y - A_yB_x, \tag{7.25}$$

and this is just the determinant for C_z given in Eqs. 7.24.

Several important problems are easily solved using the cross product. The attitude of a plane, as represented by its pole vector $\mathbf{P}$, can be obtained directly from two apparent dip vectors $\mathbf{A}_1$ and $\mathbf{A}_2$. This is written as

$$\mathbf{P} = \mathbf{A}_1 \times \mathbf{A}_2. \tag{7.26}$$

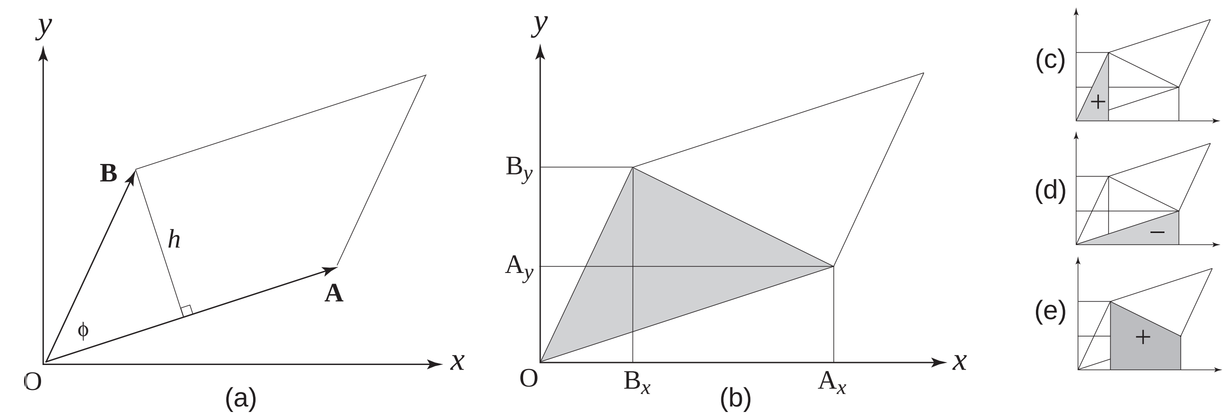

Figure 7.9: Area of the parallelogram from $\mathbf{A} \times \mathbf{B}$.

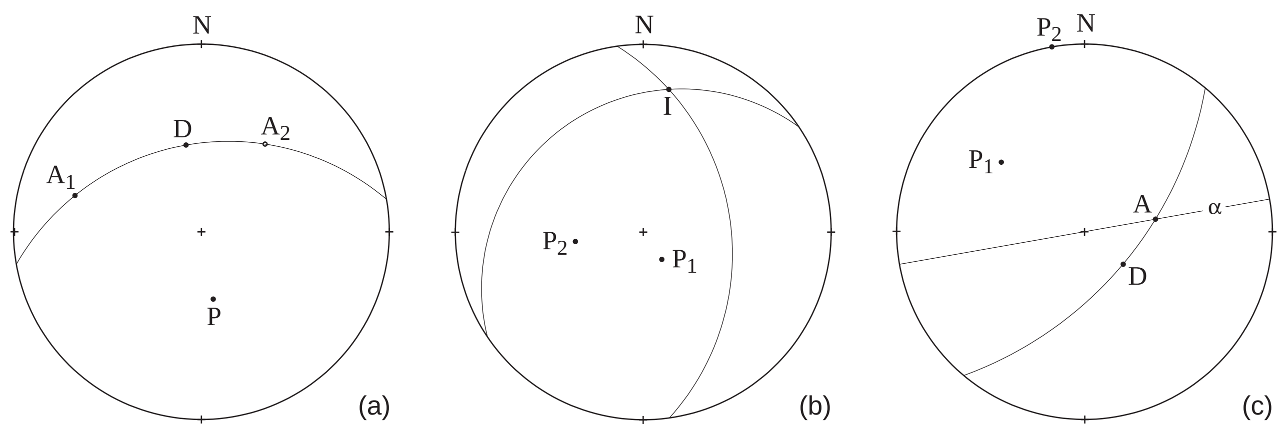

Figure 7.10: Cross product: (a) pole of a plane $\mathbf{P}$; (b) line of intersection $\mathbf{I}$; (c) apparent dip α.

Problem

- From apparent dip vectors $\mathbf{A}_1(20/286)$ and $\mathbf{A}_2(30/036)$ determine the attitude of the plane (Fig. 7.10a).

Solution

1. From the plunge and trend of each apparent dip vector, the two sets of direction cosines are

$$\mathbf{A}_1(0.30593, -0.88850, 0.34202) \quad \text{and} \quad \mathbf{A}_2(0.70063, 0.50904, 0.50000).$$

2. Perform the multiplication and then normalized components of the resulting pole vector are $\mathbf{P}(-0.61835, -0.08666, 0.77824)$.

Answer

- The plunge and trend of the pole is $\mathbf{P}(50/170)$; the attitude of the dip vector is therefore $\mathbf{D}(40/350)$.

In these types of problems it is convenient to choose the order, as we have here, so that the product vector points downward. If the reverse order is taken it will be immediately signaled by $P_z < 0$ or $n < 0$. This upward-pointing vector can be converted to the equivalent downward-pointing one by changing the signs of all three direction cosines or by changing the sign of the plunge and adding $180°$ to the trend.

The same procedure can be used to determine the orientation of the line of intersection of two planes. The intersection vector is given by

$$\mathbf{I} = \mathbf{P}_1 \times \mathbf{P}_2. \tag{7.27}$$

Problem

- From two pole vectors $\mathbf{P}_1(70/146)$ and $\mathbf{P}_2(50/262)$ determine the line of intersection of the two planes (Fig. 7.10b).

Solution

1. The components are $\mathbf{P}_1(-0.26200, 0.21985, 0.93969)$ and $\mathbf{P}_2(-0.08946, -0.63653, 0.76604)$.
2. The normalized components of the intersection vector are $\mathbf{I}(0.96122, -0.14626, 0.23379)$.

Answer

- The attitude of the line of intersection is $\mathbf{I}(14/009)$.

The cross product can also be use to find the apparent dip in a specified direction. The line of apparent dip is the intersection of the inclined plane and the vertical plane whose trend is in the apparent dip direction. From the poles of these two planes

$$\mathbf{A} = \mathbf{P}_1 \times \mathbf{P}_2, \tag{7.28}$$

where one of the poles is that of a vertical plane which contains the required direction.

Problem

- Find the apparent dip $\mathbf{A}$ in the direction 080 from the dip vector $\mathbf{D}(60/130)$ (Fig. 7.10c).

Solution

1. The pole of the inclined plane is $\mathbf{P}_1(30/310)$, and the pole of the vertical plane containing $\mathbf{A}$ is $\mathbf{P}_2(00/350)$. The two sets of direction cosines are then

$$\mathbf{P}_1(0.55667, -0.66341, 0.50000) \quad \text{and} \quad \mathbf{P}_2(0.98481, -0.17365, 0.00000).$$

2. From the cross product, after normalization, we have $\mathbf{A}(0.11604, 0.65807, 0.74396)$.

Answer

- The plunge and trend is $\mathbf{A}(48/080)$ and the plunge angle is the required apparent dip.

7.4 CIRCULAR DISTRIBUTIONS

The statistical treatment of orientation data relies heavily on vector methods. We first treat the two-dimensional case. Cheeney (1983, p. 22–26, 93f) and Middleton (2000, p. 161–167) give good introductions to the subject and the book by Fisher (1993) contains a comprehensive treatment. The use of a spreadsheet is a convenient way to manipulate such data (Tolson & Correa-Mora, 1996).

By way of introduction, we illustrate several problems associated with determining the mean direction of measured strike lines. To do this we use a small invented data set (see Table 7.1).

1. The northeast trending lines of strikes (Column A of Table 7.1) are represented by points on the circumference of a unit circle (Fig. 7.11a). A straight-forward calculation of the arithmetic mean gives the correct value of 025, that is, N 25 E (shown by the filled circle).

2. Because strike lines are axes, the trend of either end is an equally valid statement of orientation. Column B of Table 7.1 gives the same data with one trend reversed (Fig. 7.11b). Now the calculated mean of 061 is not correct.

3. The five strike lines are rotated $30°$ anticlockwise ($30°$ subtracted from each strike direction (Column C of Table 7.1), and plotted as vectors (Fig. 7.11c). Again, the arithmetic mean of -005, that is N 5 W, is correct.

4. Trends are not commonly given by negative angles; azimuths are more appropriate (Column D of Table 7.1). The mean of these gives the wildly erroneous trend of 221.

The representing of horizontal vectors by points on the circumference of a circle of unit radius may display a wide variety of forms, including uniform, unimodal, bimodal and multimodal patterns. Here we confine our treatment to the simple but important case of a single cluster, that is with a *unimodal* distribution, and the determination of its mean direction.

	A	B	C	D
1	005	005	−025	335
2	015	015	−015	354
3	025	205	−005	355
4	035	035	005	005
5	045	045	015	015
Mean	025	061	−005	211

Table 7.1: The mean direction of measured strike lines.

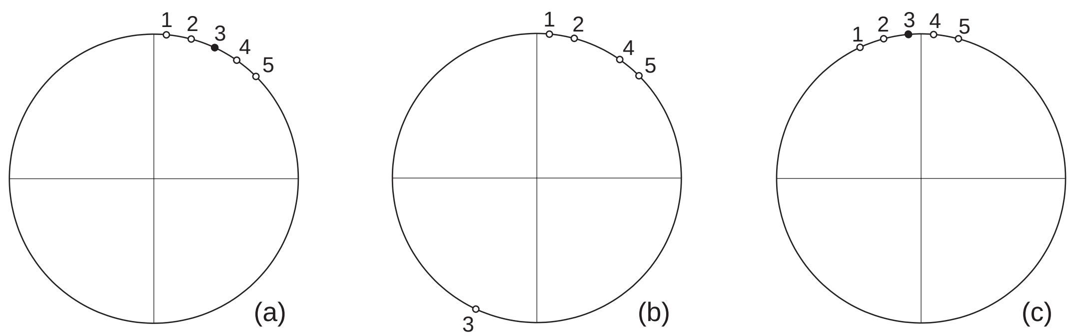

Figure 7.11: Lines of strike: (a) lines as vectors; (b) lines as axes; (c) lines rotated anticlockwise $30°$.

As we have seen the arithmetic mean of trend angles expressed as azimuths generally gives erroneous results. The reason is simple: consider two vectors with trends of 350 and 010. Clearly, the true mean direction is due north, but their arithmetic mean is $180°$ or due south.

The correct way to combine a collection of N unit vectors is by vector addition, and we do this by summing their components (Fig. 7.12).

$$C = \sum_{i=1}^{N} \cos\theta_i \quad \text{and } S = \sum_{i=1}^{N} \sin\theta_i, \tag{7.29}$$

where the $\theta_i\,(i = 1, 2\ldots, N)$ are the orientation angles of the individual vectors. The magnitude of the *resultant vector* $\mathbf{R}$ is given by

$$R = \sqrt{C^2 + S^2}, \quad (0 \le R \le N). \tag{7.30}$$

Alternatively, the mean resultant length $\bar{R}$ is

$$\bar{R} = R/N, \quad (0 \le \bar{R} \le 1). \tag{7.31}$$

$\bar{R} = 1$ implies that all points are coincident and $\bar{R} = 0$ implies a uniform distribution, but only if the data comprises a single group. The orientation of $\mathbf{R}$, which is the *mean direction*, is given by

$$\bar{\theta} = \arctan S/C. \tag{7.32}$$

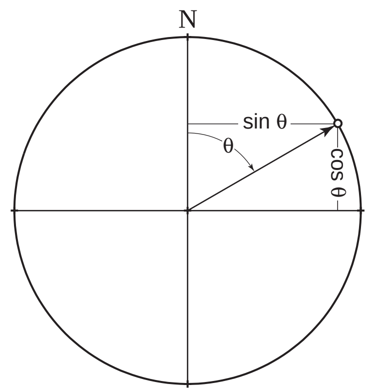

Figure 7.12: Components of trend vectors.

i	θ	$\sin\theta_i$	$\cos\theta_i$
1	245	-0.90631	-0.42262
2	254	-0.96126	-0.27564
3	272	-0.99939	0.03490
4	277	-0.99255	0.12187
5	281	-0.98163	0.19081
6	294	-0.91355	0.40674
7	301	-0.85717	0.51504
8	315	-0.70711	0.70711
9	329	-0.51504	0.85717
10	334	-0.43837	0.89879
Sums		-8.27236	3.03417

Table 7.2: Calculation of the mean of the trends of two-dimensional vectors.

As we have also seen, axial data presents another problem: because the ends of axes are interchangeable there is an inherent ambiguity. The solution is to convert the axes to true vectors by doubling the orientation angles (Krumbein, 1939; Pincus, 1956), which are now given by $2\theta \pmod{360}$ (Fisher, 1993, p. 37).[5]

Problem

- From 10 measured azimuths of the long axes of beach pebbles, determine the mean trend (Table 7.2).

[5]In modular arithmetic the expression $m \pmod{n}$ gives the remainder after integer division of m by the modulus n; for example, $466 \pmod{360} = 106$. This is sometimes called *clock arithmetic* by analogy with arithmetic on a clock face which has a modulus of $n = 12$.

Method

1. A plot using azimuths in the range 0–$360°$ shows that the trend angles lie in two distinct groups: 7 in the NE quadrant and 3 in the SW quadrant (Fig. 7.13a).

2. By doubling the orientation angles and representing each resulting vector as a point on the circumference of a unit circle they now form a single group with a range 0–$180°$ (Fig. 7.13b).

3. From each transformed trend angle $2\theta_i$, compute the values of $\cos 2\theta$ and $\sin 2\theta$ for each vector. The sums are then $C = 8.25738$ and $S = 3.29817$.

4. From Eq. 7.29 we then have components of the resultant vector $\mathbf{R}(0.26515, 0.81756)$. Then from Eqs. 7.30, 7.31, $\bar{R} = 0.86$ which also indicates a fairly strong concentration.

5. With Eq. 7.32 we have $2\bar{\theta} = 72.03113°$ or $\bar{\theta} = 36°$, and this is the mean orientation of the axes.

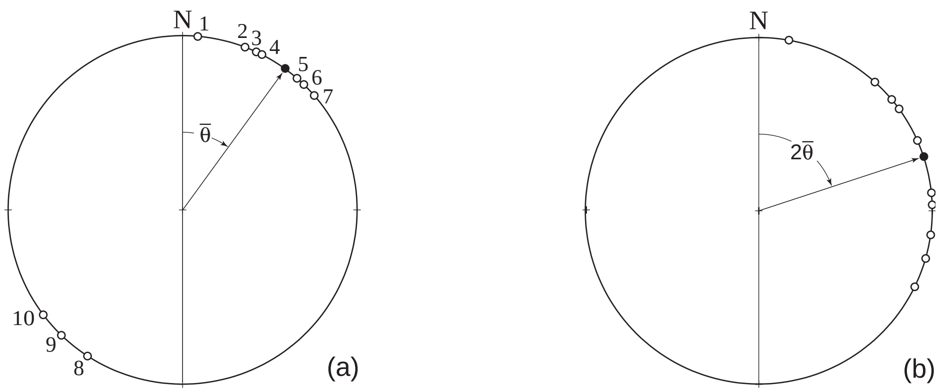

Figure 7.13: Circular distributions of axes: (a) plot; (b) conversion to vectors.

The size of the samples in this illustrative problem is small; even a single additional point might be expected to change the mean direction, possibly significantly. In practice, more measurements are needed for greater confidence.

If the analysis of problems such as these is to be extended to other statistical attributes and tests we need to take into account the entire population from which the sample was taken and in particular on the way the data points representing this population are distributed on the circle. The *circular normal* or *von Mises distribution*[6] is the most useful way of treating points which tend to cluster symmetrically about a single point. With this, a number of useful properties of such unimodal distributions can be found, but these matters would take us well beyond the level of this book. Cheeney (1983, p. 98–106) gives an easily followed discussion.

[6]This distribution is named for its formulator, the Austrian mathematician Richard von Mises [1883–1953], younger brother of the respected economist Ludwig von Mises.

7.5 SPHERICAL DISTRIBUTIONS

The extension to three dimensions is straight forward. Both Cheeney (1983, p. 107f) and Middleton (2000, p. 167–180) give good introductions and the books by Mardia (1972), Watson (1983), Fisher, et al. (1987) and Mardia & Jupp (2000) contain advanced treatments.

Three-dimensional orientation data are represented by points on a unit sphere. As in the two-dimensional case, such a collection of points can display uniform, unimodal, bimodal and girdle patterns (Mardia, 1972, p. 222; Mardia & Jupp, 2000, p. 161). We return to some related matters in Chapter 18.

As in the two-dimensional case, we treat a simple but important problem involving the distribution of points in a single cluster to illustrate the basic approach. If the cluster is approximately equidimensional it is said to be *unimodal* and the mean direction is given by the *resultant vector* $\mathbf{R}$ of the N unit vectors. Its components are

$$R_x = \sum_{i=1}^{N} l_i, \quad R_y = \sum_{i=1}^{N} m_i, \quad R_z = \sum_{i=1}^{N} n_i, \tag{7.33}$$

where the $(l_i, m_i, n_i), i = 1, 2 \ldots, N$ are the direction cosines of the individual vectors. The *resultant length* or magnitude of this vector is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}, \tag{7.34}$$

and its direction cosines are

$$\bar{l} = R_x/R, \quad \bar{m} = R_y/R, \quad \bar{n} = R_z/R. \tag{7.35}$$

R is also a measure of the concentration of the points about the mean. It will be nearly as large as N if the points are tightly clustered and will be smaller if they are dispersed. If data sets with different numbers of measurements are to be compared, the *mean resultant length* $\bar{R}$ is a more useful measure. This is defined as

$$\bar{R} = R/N, \text{ where } 0 \leq R \leq N \text{ and } 0 \leq \bar{R} \leq 1. \tag{7.36}$$

Problem

- From 10 measured poles of bedding, determine the mean attitude (Table 7.4).

Method

1. Convert the plunge p and trend t of each pole to direction cosines (l, m, n) and calculate the totals. From Eq. 7.33 we then have vector $\mathbf{R}(-6.38809, -3.67384, 6.24920)$,
2. From Eq. 7.34, $R = 9.66216$ and from Eqs. 7.35 the direction cosines of $\mathbf{R}$ are then

$$\bar{l} = -0.66115, \quad \bar{m} = -0.38023, \quad \bar{n} = 0.64677.$$

i	p	t	l_i	m_i	n_i
1	32	206	-0.76222	-0.37176	0.52992
2	30	220	-0.66341	-0.55667	0.50000
3	46	204	-0.63460	-0.28254	0.71934
4	40	198	-0.72855	-0.23672	0.64279
5	20	200	-0.88302	-0.32139	0.34202
6	32	188	-0.83979	-0.11803	0.52992
7	54	192	-0.57494	-0.12221	0.80902
8	56	228	-0.37417	-0.41556	0.82904
9	36	236	-0.45240	-0.67071	0.58779
10	44	218	-0.56685	-0.44287	0.69466
Sums			-6.38809	-3.67384	6.24920

Table 7.3: Calculation of the mean of three-dimensional vectors.

3. The attitude of the mean is $\mathbf{R}(40/210)$ (shown as a open diamond in Fig. 7.14).
4. With $R = 9.7$ and $\bar{R} = 0.97$ the points are tightly clustered about the mean, as can be seen.

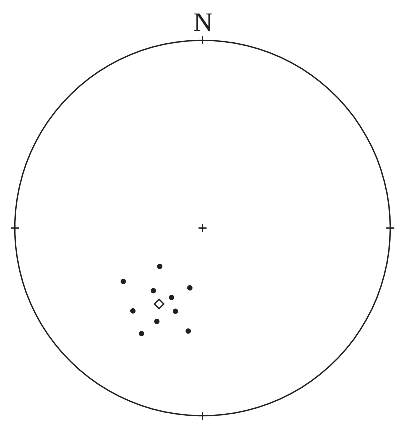

Figure 7.14: Unimodal distribution of poles and its mean.

As in the previous example problem of a circular distribution, in practice a larger number of measurements will increase confidence.

It should also be noted that the mean direction of any collection of points on the sphere may be calculated with this method, but in many situations this direction will have little or no geometrical meaning. For example, if the data are approximately uniformly distributed on the sphere, the mean vector may have almost any orientation.

If the analysis of this problem is to be extended to other statistical attributes, such as confidence limits, we need to take into account the entire population from which the sample was taken and in particular on the way the data points representing this population are distributed on the sphere. Because poles of bedding can be considered to be true vectors, the *spherical normal* or *Fisher* *distribution*[7] is the most useful way of treating points which tend to cluster symmetrically about a single point. With this, a number of useful properties of such unimodal distributions can be found, but to pursue these matters would take us well beyond the level of this book. Cheeney (1983, p. 112) and Middleton (2000, p. 167–180) give easily followed discussions.[8]

7.6 ROTATIONS

The rotations performed graphically on the stereonet can also be accomplished analytically. To do this we need expressions which relate the initial vector $\mathbf{r}(x, y, z)$ and final vector $\mathbf{r}'(x', y', z')$ in terms of an axis and angle of rotation.

Just as rotations on the stereonet may be performed simply and easily about horizontal and vertical axes, so too is it easy to describe rotations about the three axes of our coordinate system. With these descriptions we may then develop procedures for the more general cases.

Before starting we need a sign convention for the sense of a rotation about an axis and we use the *right-hand rule* — when the thumb of the right hand points in the positive direction of an axis, the fingers indicate the sense of a positive rotation.

Expressions for the rotation of a position vector $\mathbf{r}$ about the $+x$ axis are obtained from a view of the vertical yz plane looking north, that is, in the direction of $+x$ (Fig. 7.15a). Rotating about this axis, the x component remains unchanged, that is, $x' = x$, but the y and z components do change. In this plane, the orientation of $\mathbf{r}$ is given by the angle θ measured from $+y$ and the orientation of $\mathbf{r}'$ is given by the angle $\theta + \omega_x$ also measured from $+y$. Note that the length of the vector is unchanged by rotation, that is, $r = r'$. Then

$$\cos\theta = y/r \quad \text{and} \quad \cos(\theta + \omega_x) = y'/r,$$
$$\sin\theta = z/r \quad \text{and} \quad \sin(\theta + \omega_x) = z'/r.$$

Substituting these into the identities for the cosine and sine of the sum of two angles

$$\cos(\theta + \omega) = \cos\theta\cos\omega - \sin\theta\sin\omega \quad \text{and} \quad \sin(\theta + \omega) = \sin\theta\cos\omega + \cos\theta\sin\omega \tag{7.37}$$

and multiplying through by r yields expressions for y' and z'. These, plus the equality $x = x'$, are

$$\begin{aligned} x' &= x, \\ y' &= y\cos\omega_x - z\sin\omega_x, \\ z' &= y\sin\omega_x + z\cos\omega_x. \end{aligned}$$

With these equations we may obtain the components of the rotated vector from initial components x, y, z and ω_x by simple substitution. We may also represent the rotation represented by these three

[7] This distribution is named for its originator, the celebrated English statistician Ronald A. Fisher [1890–1962]. He published the description of this distribution in response to the needs of paleomagnetic studies which were then in their infancy, and it has been used extensively for this purpose ever since (see Fisher, 1953).

[8] Smith (1994) describes an interesting way of using the sphere as a tool to teach some additional and important statistical concepts to geology students.

equations with the matrix equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & -\sin\omega_x \\ 0 & \sin\omega_x & \cos\omega_x \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{7.38}$$

where the vectors $\mathbf{r}(x, y, z)$ and $\mathbf{r}'(x', y', z')$ are represented by column matrices and the rotation by the 3×3 square matrix. An important advantage of this type representation is that we can now think of the square matrix as a vector processor which changes one vector into another, and this focuses our attention on the entities rather than on their components. Such representations and their manipulation by matrix algebra have compelling advantages for many closely related problems in structural geology. The book by Ferguson (1994) gives a good introductory treatment for geology students. We illustrate a few simple applications here and in several later chapters.

The three algebraic equations can be obtained directly from the matrix equation of Eq. 7.38. To do this, we think of each row of the square matrix as a vector. Then *row times column multiplication* corresponds to finding the dot product of each row and the column vector (see Eq. 7.15). The basic method follows an easily remembered pattern. Consider the first row of the square matrix and ignore the other two. We then have

$$\begin{bmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} au + bv + cw \\ \cdot \\ \cdot \end{bmatrix}. \tag{7.39a}$$

The second element of the resulting column vector is obtained in the same way by forming the dot product using the second row of the square matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ d & e & f \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cdot \\ du + ev + fw \\ \cdot \end{bmatrix}, \tag{7.39b}$$

and finally, the third element of the resulting column vector is the dot product using the third row

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g & h & i \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ gu + hv + iw \end{bmatrix}. \tag{7.39c}$$

With a little practice the pattern of making each of these combinations becomes automatic. In forming the three dot products it helps to keep track of the each product by stepping along each of the rows with the left index finger while stepping down the column with the right index finger.

For a rotation about the y axis we obtain expressions for the changes in the x and z components on the vertical xz plane looking east, that is, in the $+y$ direction (Fig. 7.15b). In this plane the orientation of $\mathbf{r}$ is given by the angle θ it makes with the $+x$ axis and the orientation of $\mathbf{r}'$ by the angle ω_y it makes with $\mathbf{r}$. Then

$$\cos\theta = x/r \quad \text{and} \quad \cos(\theta - \omega_z) = x'/r,$$
$$\sin\theta = z/r \quad \text{and} \quad \sin(\theta - \omega_z) = z'/r.$$

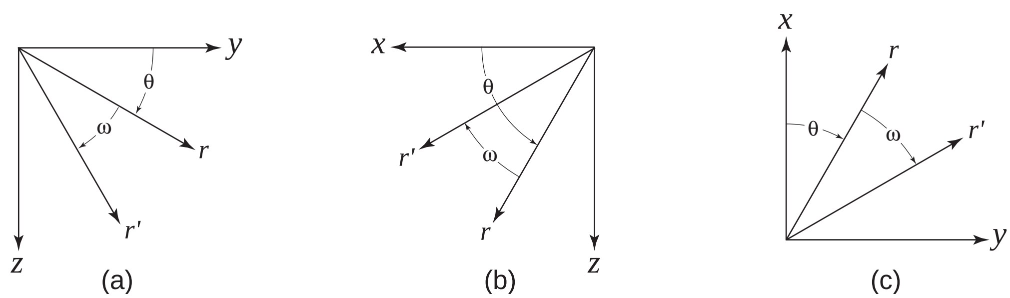

Figure 7.15: Positive rotations: (a) about $+x$; (b) about $+y$; (c) about $+z$.

Using these expressions in the identities for the cosine and sine of the difference of two angles

$$\cos(\theta - \omega) = \cos\theta\cos\omega + \sin\theta\cos\omega \quad \text{and} \quad \sin(\theta - \omega) = \sin\theta\cos\omega - \cos\theta\cos\omega, \tag{7.40}$$

yields the three equations

$$\begin{aligned} x' &= x\cos\omega_y + z\sin\omega_y, \\ y' &= y, \\ z' &= -x\sin\omega_y + z\cos\omega_y. \end{aligned}$$

In matrix form this rotation is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\omega_y & 0 & \sin\omega_y \\ 0 & 1 & 0 \\ -\sin\omega_y & 0 & \cos\omega_y \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{7.41}$$

Finally, the rotation about the z axis is described on the horizontal xy plane looking down (Fig. 7.15c). In this plane the orientation of $\mathbf{r}$ is given by the angle θ it makes with the $+x$ axis and the orientation of $\mathbf{r}'$ by the angle ω_z it makes with $\mathbf{r}$. Then

$$\begin{aligned} \cos\theta = x/r \quad &\text{and} \quad \cos(\theta + \omega_z) = x'/r, \\ \sin\theta = y/r \quad &\text{and} \quad \sin(\theta + \omega_z) = y'/r. \end{aligned}$$

From the identities of Eqs. 7.37 we have the three equations

$$\begin{aligned} x' &= x\cos\omega_z - y\sin\omega_z, \\ y' &= x\sin\omega_z + y\cos\omega_z, \\ z' &= z, \end{aligned}$$

or the single matrix equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\omega_z & -\sin\omega_z & 0 \\ \sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{7.42}$$

These results are fully general in the sense that they apply to the rotation of vectors of any magnitude. In all our examples, however, we treat only unit vectors as represented by direction cosines.

We represent each of the three square rotational matrices of Eqs. 7.38, 7.41 and 7.42 by the symbols $\mathbf{R}_x(\omega_x)$, $\mathbf{R}_y(\omega_y)$ and $\mathbf{R}_z(\omega_z)$.[9] Each of these rotations represents a corresponding graphical procedure used to rotate about vertical and horizontal axes on the stereonet. As we have indicated, each of these may be treated either as a set of three equations which can be manipulated by simple substitution or as a matrix multiplication.

We may also combine several separate rotation matrices into a single rotation matrix $\mathbf{R}$. For example, the sequence of rotations, first about $+z$ and then about $+x$, may be written in this notation as

$$\mathbf{R} = \mathbf{R}_x(\omega_x)\mathbf{R}_z(\omega_z),$$

where $\mathbf{R}_z$ is applied first and then $\mathbf{R}_x$, that is, the order is taken from *right to left*. Adhering to this order is important because finite rotations are not commutative.

The product matrix $\mathbf{R}$ represents the single equivalent rotation. With the square matrices of Eqs. 7.38 and 7.42, representing rotations about the x and z axes, we then have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & -\sin\omega_x \\ 0 & \sin\omega_x & \cos\omega_x \end{bmatrix} \begin{bmatrix} \cos\omega_z & -\sin\omega_z & 0 \\ \sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\omega_z & -\sin\omega_z & 0 \\ \cos\omega_x\sin\omega_z & \cos\omega_x\cos\omega_z & -\sin\omega_x \\ \sin\omega_x\sin\omega_z & \sin\omega_x\cos\omega_z & \cos\omega_x \end{bmatrix}. \tag{7.43}$$

The elements of this resulting 3×3 product matrix are obtained by an extension of the pattern of Eqs. 7.39 again using *row times column multiplication*. To see this more clearly focus on the first row of the left-hand matrix and the first column of the right-hand matrix, disregarding all the others. We then see only

$$\begin{bmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} p & \cdot & \cdot \\ q & \cdot & \cdot \\ r & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} ap+bq+cr & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$

The resulting element in the product matrix is the *dot product* of this row and this column. Note that this element is located in the position common to the row and column, that is, in the first row and first column.

All the elements of the product matrix are obtained in this same way: put your left index finger against any row of the square matrix on the left and your right index finger against any column of the square matrix on the right; the three pairs of elements so identified appear in a single element of the product matrix as the sum of the products of corresponding elements. The position of each product element is the one common to the selected row and column.

Problem

- Rotate line $\mathbf{L}(00/320)$, first with $\mathbf{R}_z(-60°)$ then with $\mathbf{R}_x(-40°)$ (see Fig. 6.5b).

[9] The symbol $\mathbf{R}$ here should not be confused with the resultant vector of the previous sections.

Solution

1. First, convert the plunge and trend of $\mathbf{L}$ into direction cosines expressed as a column matrix.
2. Then substitute the rotational angles into the single product matrix of Eq. 7.43.
3. The full equation is then

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0.76604 & 0.64279 & 0.00000 \\ -0.32139 & 0.38302 & 0.86603 \\ 0.55667 & -0.66341 & 0.50000 \end{bmatrix} \begin{bmatrix} 0.76604 \\ -0.64279 \\ 0.00000 \end{bmatrix} = \begin{bmatrix} -0.17365 \\ -0.49240 \\ 0.85287 \end{bmatrix}.$$

Answer

- After the combined rotations, the attitude is $\mathbf{L}'(59/289)$ and this is the same result obtained graphically.

This same procedure can be extended to any number of rotations. By hand such multiple rotations require a tedious series of computations but the sequence can be easily programmed.

7.7 REFLECTION

As we have seen, there are two different sets of Cartesian coordinate axes in use: one for orientation data on the stereonet(Fig. 7.16a$_1$) and the other for geographical data (Fig. 7.16a$_2$). It is sometimes necessary to convert from one of these to the other. Comparing unit vectors in each of the positive directions we can write down these changes as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

With these three resulting columns we can write down the transformation of (x, y, z) to (x', y', z'), where they appear in order as the three columns in the 3×3 matrix

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In a closely related transformation, it is sometimes necessary to convert data plotted on the xy plane of the right-handed set of axes (Fig. 7.16b$_1$) to the $x'y'$ plane of the left-handed axes set (Fig. 7.16b$_2$). For example, if data plotted on the stereonet or related projections, we may wish to convert it to second set of axes to take advantage of the standard plotting routines available in graphic programs. As before, we write down the conversion of the unit vectors in the first set to the same vectors in the second set. We write these as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The full transformation is then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This special transformation does *not* describe a rotation, but a *reflection*. Such transformations, while not applicable to the rotation of physical entities, it does have a role in a number of other applications, including the description of the symmetry properties of crystals. Its special property is that it describes a refection about the sloping lines $x = y$ and $x' = y'$.

There is a simple calculation which can be used to distinguish between matrices that describe rotations and reflections: it is the *determinant* (see Figs. 7.7.8b and 7.21). For an orthogonal matrix describing a rotation $\det = +1$ and for a reflection $\det = -1$.

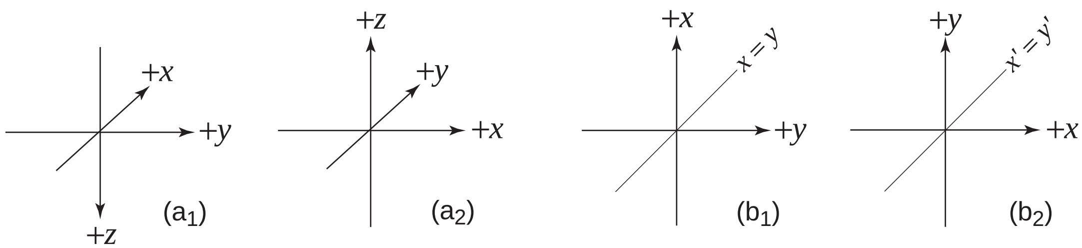

Figure 7.16: Coordinate axes: (a) set of three-dimensional axes; (b) set of two-dimensional axes .

7.8 ROTATIONAL PROBLEMS

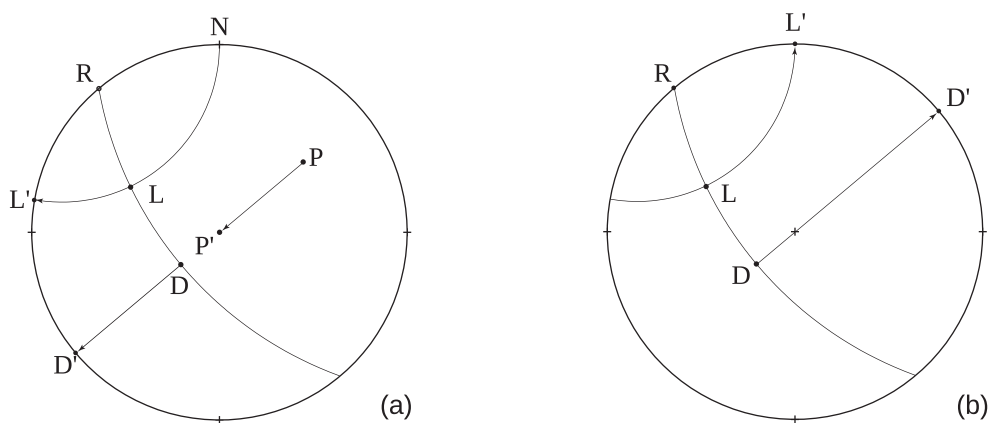

Figure 7.17: Conventional tilt correction: (a) upright; (b) overturned.

With these matrix representations of rotations about the coordinate axes we can solve all the rotational problems of the previous chapter. As we have just seen any sequence of rotations can be combined by matrix multiplication into an equivalent single rotation matrix that produces the same result.

A simple but important geologic problem is the restoration of the pre-tilt orientation of a line in an inclined plane by the conventional tilt correction. Following the procedure used in Fig. 6.8, we specify the attitude of the plane by the plunge and trend of the dip vector $\mathbf{D}$. To form the rotation matrix which restores the plane to horizontality by rotation about the strike direction then requires three steps.

Steps

1. Rotate vector $\mathbf{D}$ about $+z$ into the vertical xz plane by $\mathbf{R}_z(-t)$.
2. Rotate $\mathbf{D}$ about $+y$ to horizontal by $\mathbf{R}_y(\delta)$.
3. Return $\mathbf{D}$ its original trend by $\mathbf{R}_z(+t)$.

This sequence can be represented by the equation

$$\mathbf{R}(\omega) = \mathbf{R}_z(+t)\,\mathbf{R}_y(\delta)\,\mathbf{R}_z(-t), \tag{7.44}$$

where again the order is taken from right to left. If the bed is overturned then the rotation about y is given by $\mathbf{R}_y(\delta - 180°)$.

Problem

- A plane whose attitude is given by $\mathbf{D}(60/230)$ contains line $\mathbf{L}$ with $t = 297°$. What was the pre-tilt trend of the line? (Fig. 7.17a; see also Fig. 6.8a).

Solution

1. The single equivalent rotation is found from the sequence of rotations

$$\mathbf{R} = \mathbf{R}_z(-230°)\,\mathbf{R}_y(+60°)\,\mathbf{R}_z(+230°).$$

2. Using Eq. 1.8, the angle the trend of $\mathbf{L}$ makes with the dip direction is $\phi = 297 - 230 = 67°$. Then the plunge of this line is $\alpha = 34.08881°$.
3. Its plunge and trend give $\mathbf{L}(0.37598, -0.73790, 0.56048)$.
4. The full rotation matrix equation is then

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0.79341 & -0.24620 & -0.55667 \\ -0.24620 & 0.70659 & -0.66341 \\ 0.55667 & 0.66341 & 0.50000 \end{bmatrix} \begin{bmatrix} 0.37598 \\ -0.73790 \\ 0.56048 \end{bmatrix} = \begin{bmatrix} 0.51593 \\ -0.85663 \\ 0.00000 \end{bmatrix}.$$

Answer

- The estimated pre-tilt attitude of the line is $\mathbf{L}'(00/280)$.

Problem

- If the beds in the previous problem are overturned what was the pre-tilt trend of the line? (Fig. 7.17b; see also Fig. 6.8b).

Solution

1. The single equivalent rotation is found from the sequence

$$\mathbf{R} = \mathbf{R}_z(-230^\circ)\,\mathbf{R}_y(-120^\circ)\,\mathbf{R}_z(+230^\circ).$$

2. As before, using the given trend and Eq. 1.8, determine the plunge of the line in the plane and then its direction cosines.

3. In matrix form the set of equations representing the single rotation is then

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0.38024 & -0.73861 & 0.55667 \\ -0.73861 & 0.11976 & 0.66341 \\ -0.55667 & -0.66341 & -0.50000 \end{bmatrix} \begin{bmatrix} 0.37598 \\ -0.73790 \\ 0.56048 \end{bmatrix} = \begin{bmatrix} 0.99998 \\ 0.00575 \\ 0.00000 \end{bmatrix}.$$

Answer

- The estimated pre-tilt attitude was $\mathbf{L}'(00/000)$, that is, horizontal and trending due north.

In both these solutions the plunge of the line in the inclined plane was calculated from its trend using the apparent dip formula. If a measured plunge angle is used or its value is read from a plot it may not lie exactly in the plane and this may result in the corrected attitude departing slightly from horizontal. Even if the plunge is accurately calculated, a tiny round-off error may produce the same result. If, because of these errors, the restored line ends up in the upper hemisphere and it is reversed into the lower hemisphere the trend will be 180° in error. In such cases, some care is required when interpreting the results.

The case of a rotation about an inclined axis requires a sequence of five coordinate rotations. There are several equivalent ways of ordering these and the one we choose is closely related to the procedure used graphically in the previous chapter.

Steps

1. Rotate axis R about the $+z$ axis by angle $-t$ to bring it into the vertical xz plane.

2. Rotate this R about the $+y$ axis by angle $(p - 90°)$ to bring it into coincidence with the $+z$ axis.

3. Rotate about the $+z$ axis by the specified angle ω to perform the required rotation.

4. Rotate R about the $+y$ axis by angle $(90° - p)$ as the first step in returning it to its original orientation.

5. Finally, rotate R about the $+z$ axis by angle $+t$ to return it to its initial orientation.

We may also express this sequence of five rotations in short-hand as

$$\mathbf{R}(\omega) = \mathbf{R}_z(+t)\,\mathbf{R}_y(90° - p)\,\mathbf{R}_z(\omega)\,\mathbf{R}_y(p - 90°)\,\mathbf{R}_z(-t). \tag{7.45}$$

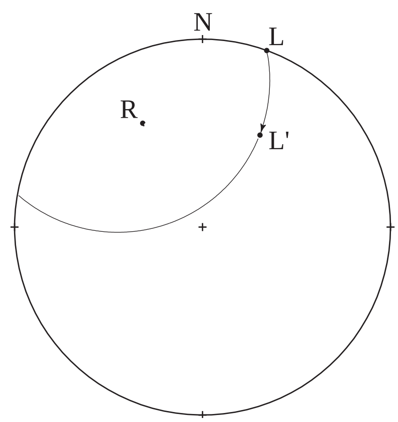

Figure 7.18: Single rotation equivalent to a sequence of rotations.

Problem

- Rotate line $L(00/020)$ about the inclined axis $R(25/330)$ by $\omega = +40°$ (Fig. 7.18; see also Fig. 6.8).

Solution

1. The complete sequence of rotations is given by

$$\mathbf{R}(\omega) = \mathbf{R}_z(330°)\,\mathbf{R}_y(65°)\,\mathbf{R}_z(40°)\,\mathbf{R}_y(-65°)\,\mathbf{R}_z(-330°). \tag{7.46}$$

2. Performing the multiplication of the five matrices, together with the direction cosines of the line, yields

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0.91017 & -0.35487 & -0.21368 \\ 0.18844 & 0.81409 & -0.54932 \\ 0.36889 & 0.45971 & 0.80783 \end{bmatrix} \begin{bmatrix} 0.93969 \\ 0.34202 \\ 0.00000 \end{bmatrix} = \begin{bmatrix} 0.73391 \\ 0.45551 \\ 0.50387 \end{bmatrix}.$$

Answer

- From these direction cosines, the plunge and trend of the line after rotation is $\mathbf{L}'(30/032)$.

7.9 TRANSFORMATION OF AXES

As we have seen in these examples, any sequence of rotations can always be written as a single matrix representing a single rotation which produces the same final result. This again illustrates Euler's theorem. As a consequence, it is not possible to recover the separate rotational steps from angular measurements of the final state alone. The best that we can hope for is the single matrix representing the total rotation.

Until now we have concentrated on the rotation of various physical lines and planes of geologic interest within a body of rock. Because it illuminates an important property of the rotation matrix, we now consider a set of unit base vectors which we take as embedded in a rigid body and which rotates with it.

In order to see the geometrical meaning of the rotation matrix more clearly it is advantageous to relabel the three coordinate axes $x_1 = x$, $x_2 = y$ and $x_3 = z$. Similarly, we introduce a notational scheme for identifying the components of the rotation matrix using numerical subscripts.[10] The rotation matrix is now written as

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{7.47}$$

where the first subscript identifies the row number and the second identifies the column number. Now the matrix equation describing the general rotation of a vector has the form

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{7.48}$$

Each element of the rotation matrix has a geometric meaning. To show this, we choose to rotate the unit vector initially in the x_1 direction, which we write as

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix}.$$

[10]The introduction of these subscripts was essential in the development of matrix algebra (Lanczos, 1954, p. 54). It made clear that the matrix elements were really components of single entities. It also brought out patterns which might be overlooked and made programming the equations easier.

Thus the elements in the first column of $\mathbf{R}$ represent the three components of the unit base vector $\mathbf{i}$ initially in the x_1 direction, that is, they are its *direction cosines*. In the same way

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}R_{12}\\R_{22}\\R_{32}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}R_{13}\\R_{23}\\R_{33}\end{bmatrix}.$$

and similarly, the second and third columns contain the components of the unit vectors $\mathbf{j}$ and $\mathbf{k}$ initially in the x_2 and x_3 directions. From these three results it can be seen that all the elements of $\mathbf{R}$ are actually direction cosines relating one set of coordinate axes to the other. For this reason the rotation matrix is sometimes written as

$$\mathbf{A} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},$$

where the elements a_{ij} are now explicitly direction cosines. For example, the direction cosines of x_1' with respect to x_1, x_2 and x_3 are (a_{11}, a_{21}, a_{31}) and the corresponding direction angles are $\alpha_{11} = \arccos a_{11}$, $\alpha_{21} = \arccos a_{21}$ and $\alpha_{31} = \arccos a_{31}$. In this scheme, the second subscript refers to the *old* axes and the first to the *new* axes.[11]

At most only three angles ω_1, ω_2, ω_3 are required to specify a fully general rotation, so the elements of $\mathbf{A}$ cannot all be independent. Because they are the components of a unit vector, the direction cosines in the first column obey the identity of Eq. 7.5, that is,

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1,$$

and similar equations also hold for the elements of the other two columns. Also, because they are perpendicular, the dot product of the elements of any two columns equals zero (Eq. 7.17). For example, from the first and second columns

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0,$$

and there are similar equations for the other two pairs of columns. In all there are six of these equations. These are the *orthogonality relations* and matrices for which these hold are termed *orthogonal*.

Problem

- Plot the three transformed axes x_1', x_2', x_3' from the orthogonal matrix (see Eq. 7.44)

$$\mathbf{A} = \begin{bmatrix}0.91017 & -0.35487 & -0.21368\\ 0.18844 & 0.81409 & -0.54932\\ 0.36889 & 0.45971 & 0.80783\end{bmatrix}.$$

[11] Be careful. Some writers reverse the meaning of the subscripts.

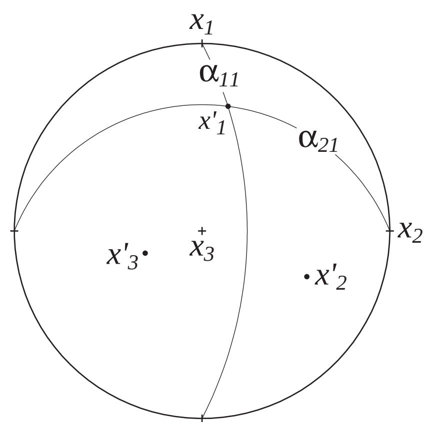

Figure 7.19: Transformation of coordinate axes in three dimensions.

Result

- With Eqs. 7.8 each column gives in turn $x_1'(22/012)$, $x_2'(27/114)$ and $x_3'(54/249)$ (Fig. 7.19).

This result again illustrates the important fact that, though it may be generated by a sequence of rotations, a single orthogonal matrix describes nothing more than the angular relationship between two sets of coordinate axes. For this reason, it is said to describe a *transformation of axes*.

In several important applications we will have to deal with how vectors and related entities behave under a transformation of axes. By a transformation of axes is meant a change from ones set of orthogonal axes to another with the same origin. In two dimensions the matrix representing such a transformation is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{7.49}$$

First, we show the application of the transformation of the components of a vector. Such transformations are presented in most introductory books on analytic geometry and calculus (e.g., Thomas & Finney, 1993, p. 641; also Eisenhart, 1966, p. 149, 208).

Starting with a vector $\mathbf{V}$ referred to a particular set of axes (Fig. 7.20a), there are two different but related approaches to the transformation of these components.

1. A fixed vector relative to rotated axes is called an *alias* because it is the same vector but with a different name (Fig. 7.20b). In this example, the sense of natation is positive.

2. The rotation of a vector relative to fixed axes is called an *alibi* because the vector is now in a different place (Fig. 7.20c). Here the rotation is negative, that is, the opposite sense.

As we will see, both formulations give the same result but because it is the standard approach we choose the first.

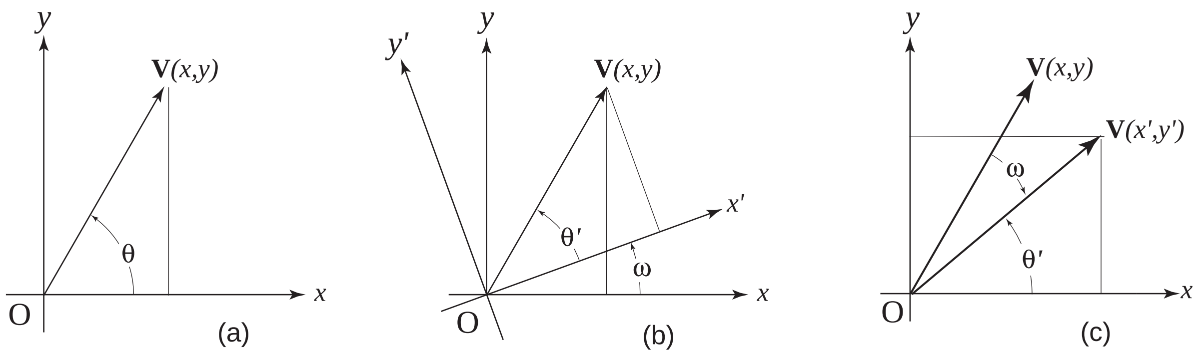

Figure 7.20: Transformation of a vectors: (a) original axes; (b) rotated axes; (c) rotated vector.

Problem

- Vector **V** has its components referred to xy axes. What will its components be when referred to transformed axes $x'y'$?

Derivation

1. If **V** makes angle θ with the x axis (Fig. 7.20a) then its components are

$$x = V\cos\theta, \tag{7.50a}$$
$$y = V\sin\theta. \tag{7.50b}$$

2. Similarly, **V** makes angle θ' with the transformed axis x' (Fig. 7.20b) and its components are

$$x' = V\cos\theta', \tag{7.51a}$$
$$y' = V\sin\theta'. \tag{7.51b}$$

3. The angles θ and θ' are related by

$$\theta = \theta' + \omega, \tag{7.52}$$

   where ω is the angle between the x and x' axes measured from x.

4. With Eq. 7.50 we then have

$$x = V\cos(\theta' + \omega),$$
$$y = V\sin(\theta' + \omega).$$

5. Substituting the identifies for the sine and cosine of the difference of two angles

$$\sin(\theta' + \omega) = \sin\theta'\cos\omega + \cos\theta'\sin\omega \quad \text{and} \quad \cos(\theta' + \omega) = \cos\theta'\cos\omega - \sin\theta'\sin\omega$$

yields

$$x = V\cos(\theta' + \omega) = V\cos\theta'\cos\omega - V\sin\theta'\sin\omega,$$
$$y = V\sin(\theta' + \omega) = V\cos\theta'\cos\omega + V\sin\theta'\cos\omega.$$

6. Then substituting Eqs. 7.51 these simplify to

$$x = x'\cos\omega - y'\sin\omega, \tag{7.53a}$$
$$y = x'\sin\omega + y'\cos\omega. \tag{7.53b}$$

7. It is useful to write this pair as the matrix equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}. \tag{7.54}$$

This expresses the (x, y) components in terms of the transformed components (x', y') and the rotation.

8. For our immediate purposes we need to recast Eq. 7.54. We do this with the matrix with the opposite sense of rotation. The simplest way is to form the *transpose*by interchanging the rows and columns. Doing this Eq. 7.49 becomes

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}. \tag{7.55}$$

By pre-multiplying (that is from the left) both sides of Eq. 7.54 by the transpose matrix gives

$$\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix},$$

the two rotations on the right-hand side cancel leaving

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{7.56}$$

subsection*Calculation

1. From Fig. 7.20b, $\theta = 60°$ and $\omega = 20°$. Using these values and taking $V = 1$ Eq. 7.54 becomes
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.9397 & 0.3420 \\ -0.3420 & 0.9397 \end{bmatrix} \begin{bmatrix} 0.5000 \\ 0.8660 \end{bmatrix} = \begin{bmatrix} 0.7660 \\ 0.6420 \end{bmatrix}.$$

2. As a check $\theta' = y'/x' = 40°$. which is the same value as required by Eq. 7.52.

A second important application is to express the equation of an ellipse in transformed axes (Fig. 7.21a). The general equation of an ellipse centered at the origin is

$$Ax^2 + 2Bxy + Cy^2 = 1. \tag{7.57}$$

Some authors write this equation using Bxy (e.g., Thomas & Finney, 1993, p. 640) while others use $2Bxy$ (e.g., Boas, 1983, p. 420). We prefer this second form because we can represent the coefficients of the ellipse as a square matrix in a simple way.[12] With this matrix the equation of the ellipse is then

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1. \tag{7.58}$$

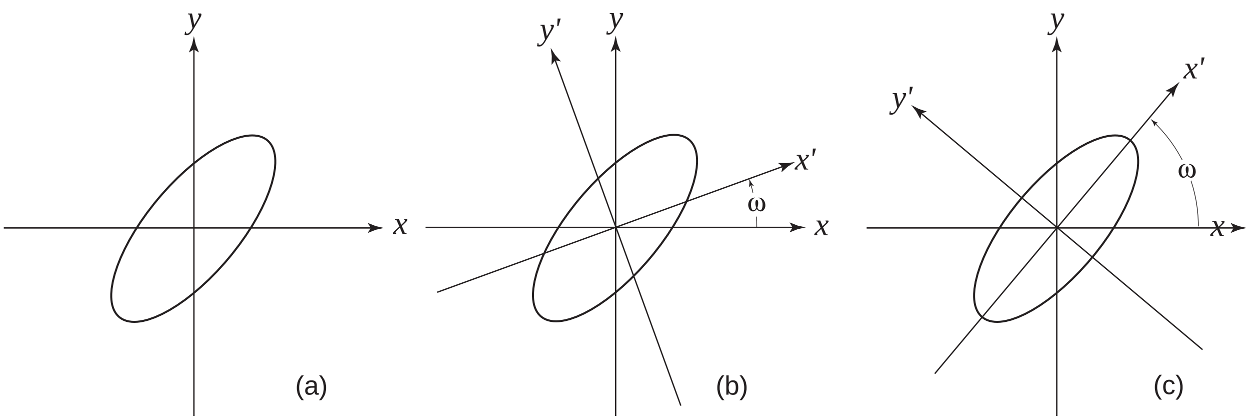

Figure 7.21: Transformation of an ellipse: (a) in original axes; (b) general transformation; (c) special transformation.

To transform this matrix equation of the ellipse requires several steps.

1. First, substitute the expression for the x, y components from Eq. 7.54 for the column matrix giving

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 1. \tag{7.59}$$

2. Next, we need to make a similar substitution for the row matrix. We do this by forming the transpose of both sides of Eq. 7.54. On the left-hand side the column matrix become a row matrix. On the right-hand side, the product of the square and column matrix employs the *reversal rule* whereby the transpose of a product is equal to the product of the transposes in reverse order. The results is

$$\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \tag{7.60}$$

[12]We will see in Chapter 12 that this square matrix of coefficients has important applications in strain analysis.

3. Substituting this in Eq. 7.58, the full result is

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 1. \tag{7.61}$$

Expanding the three square matrices yields

$$\begin{bmatrix} A' & B' \\ B' & C' \end{bmatrix} = \begin{bmatrix} A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta & (C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ (C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) & A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta \end{bmatrix}. \tag{7.62}$$

Of special interest is the case of transforming the coordinate axes to coincide with the principal axes of the ellipse (Fig. 7.21c). Referred to these axes $B = 0$, and the formula becomes

$$A'x'^2 + C'y'^2 = 1, \tag{7.63}$$

and the lengths of the semi-axes are given by $1/\sqrt{A'}$ and $1/\sqrt{C'}$.

7.10 THREE-POINT PROBLEM

The three-point problem may be solved analytically in several ways. Haneberg (1990) described a technique involving Lagrangian interpolation and De Paor (1991) used barycentric coordinates. Here we illustrate two additional vector-related ways.

Coordinate Geometry

The first uses coordinate geometry to determine the components of the vector normal to the plane. Because elevations on land are almost always positive numbers it is convenient, and universal, to adopt the right-handed system of geographic coordinate axes with $+x$ = east, $+y$ = north and $+z$ = up. Note that in contrast to our previous coordinate system *positive* vertical angles are now measured upward and *positive* horizontal angles are measured anticlockwise from $+x$.

We need the equation of the plane passing through three non-collinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, and this requires the solution of the system of homogeneous equations (see Vacher, 2000e)

$$\begin{aligned} Ax + By + Cz + D &= 0, \\ Ax_1 + By_1 + Cz_1 + D &= 0, \\ Ax_2 + By_2 + Cz_2 + D &= 0, \\ Ax_3 + By_3 + Cz_3 + D &= 0. \end{aligned}$$

The first of these is the general form of the equation of the plane. The other three express the conditions that the three points lie on this plane. We may also write these in the form of a matrix equation

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This equation always has the trivial solution $A = B = C = D = 0$, but this has no physical meaning. A non-trivial solution exists if and only if the determinant of the 4×4 matrix is equal to zero

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Expanding by the method of cofactors gives the required equation of the plane

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

There are two ways of expanding these 3×3 determinants.

1. If there are all 1s in any column, as in the first three terms, the method of cofactors is particularly easy to apply.

2. In the more general case a simple extension of the method used for a 2×2 determinant is perhaps the easiest approach. Copy the first two columns to the right. Then the three triple products from the upper left to lower right are positive (Fig. 7.22a) and the three triple products from upper right to lower left are negative (Fig. 7.22b).

Applying these yields

$$A = + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = +[(y_2 z_3 - z_2 y_3) - (y_1 z_3 - z_1 y_3) + (y_1 z_2 - z_1 y_2)], \tag{7.64a}$$

$$B = - \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = -[(x_2 z_3 - z_2 x_3) - (x_1 z_3 - z_1 x_3) + (x_1 z_2 - z_1 x_2)], \tag{7.64b}$$

$$C = + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = +[(x_2 y_3 - y_2 x_3) - (x_1 y_3 - y_1 x_3) + (x_1 y_2 - y_1 x_2)], \tag{7.64c}$$

$$D = - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = -[(x_1 y_2 z_3 + y_1 z_2 x_3 + z_1 x_2 y_3) - (z_1 y_2 x_3 + x_1 z_2 y_3 + y_1 x_2 z_3)]. \tag{7.64d}$$

Geometrically, the coefficients A, B and C are the components of a vector $\mathbf{N}$ normal to the plane. The constant D is related to the distance from the origin to the plane in this direction (which we don't need in this application). With A, B and C evaluated, the equation of the normal vector is then

$$\mathbf{N} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}. \tag{7.65}$$

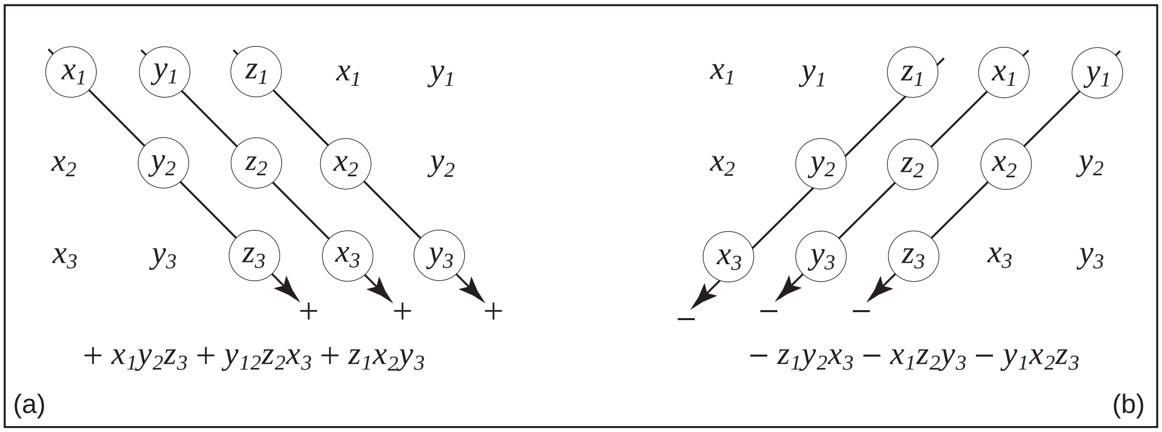

Figure 7.22: Evaluating a 3×3 determinant.

	x	y	z
P_1	100 m	60 m	535 m
P_2	350 m	16 m	415 m
P_3	156 m	214 m	440 m

Table 7.4: Dip and strike from coordinate geometry.

Problem

- From points P_1, P_2 and P_3 on a plane, determine its attitude (see Fig. 7.23 and Table 7.4).

Solution

1. From Eqs. 7.59 the values of the coefficients are $A = 22660$, $B = 17030$ and $C = 40964$. These are direction numbers of the vector $\mathbf{N}$ normal to the plane.

2. Normalizing these numbers, the direction cosines are $\mathbf{N}(0.45488, 0.34186, 0.82232)$. Note that because the direction of $+z$ is taken *upward*, these represents an *upward* pointing vector which is plotted in the upper hemisphere in Fig. 7.24a.

3. The opposite of $\mathbf{N}$ is the downward pointing pole vector $\mathbf{P}$, that is, $\mathbf{P} = -\mathbf{N}$. This vector is plotted in the lower hemisphere in Fig. 7.24b.

4. The trend of the dip vector $\mathbf{D}$ and the trend of $\mathbf{N}$ are the same.

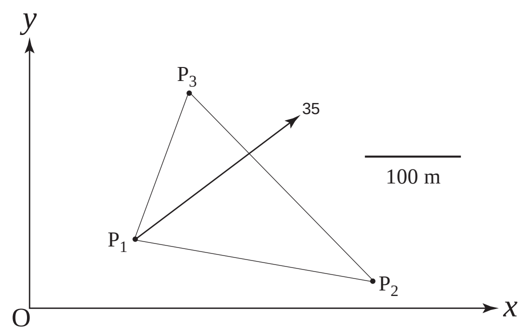

Figure 7.23: Three-point problem by coordinate geometry.

Answer

- Using Eqs. 7.8 gives the correct plunge of the dip vector but its trend is measured from $+x =$ east. We need its complement and we thus have $\mathbf{D}(35/053)$.

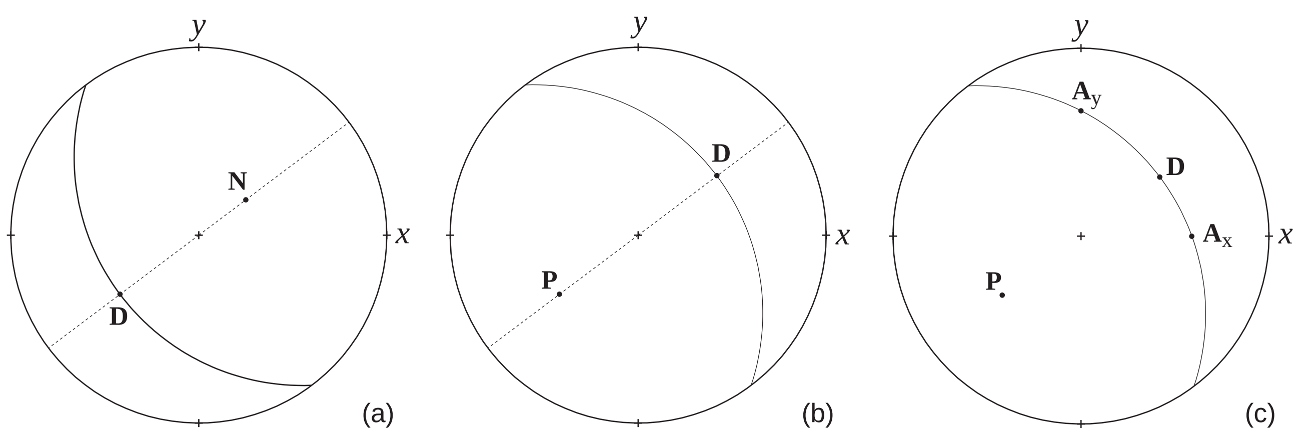

Figure 7.24: Three-point problem: (a) $\mathbf{N}$ and $\mathbf{D}$ in the upper hemisphere; (b) $\mathbf{P}$ and $\mathbf{D}$ in the lower hemisphere; (c) true dip $\mathbf{D}$ from apparent dips $\mathbf{A}_x$ and $\mathbf{A}_y$.

Vector Analysis

The second method uses elementary vector analysis. Not only does this provide a simple solution but the basic approach is directly applicable to a wide variety of other physical problems. The treatment closely follows Vacher (1989).

Associated with every point on a map depicting the inclined plane is a number representing its height h. The functional relationship between the elevation and these geographic points is written $h(x, y)$.

In mathematical terms $h(x, y)$ is a *two-dimensional scalar field*. At every point in this field the rate of change of h with distance s depends on direction. This is the *directional derivative* and it is denoted dh/ds. The difference Δh in the heights between any two points on the inclined plane is found from the slope and map distance between the points, that is,

$$\Delta h = (dh/ds)s.$$

There is a direction in which dh/ds has a maximum value, and this direction of maximum slope is represented by a vector called the *gradient* of h, written grad h or $\boldsymbol{\nabla} h$.[13] In component form this gradient vector is given by the sum of the vector components in each of the coordinate directions.

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit base vectors in the $+x$ and $+y$ directions and the partial derivatives $\partial h/\partial x$ and $\partial h/\partial y$ are the slopes of the plane in each of these directions. We can now express the *directional derivative* in any direction as the dot product

$$\boldsymbol{\nabla} h \cdot \hat{\mathbf{u}} = \frac{dh}{ds},$$

where $\hat{\mathbf{u}}$ is the unit vector in the required direction.

This gradient vector exist at every point in the field, expressed as $\boldsymbol{\nabla} h(x, y)$, and this is the description of a *vector field*.[14]

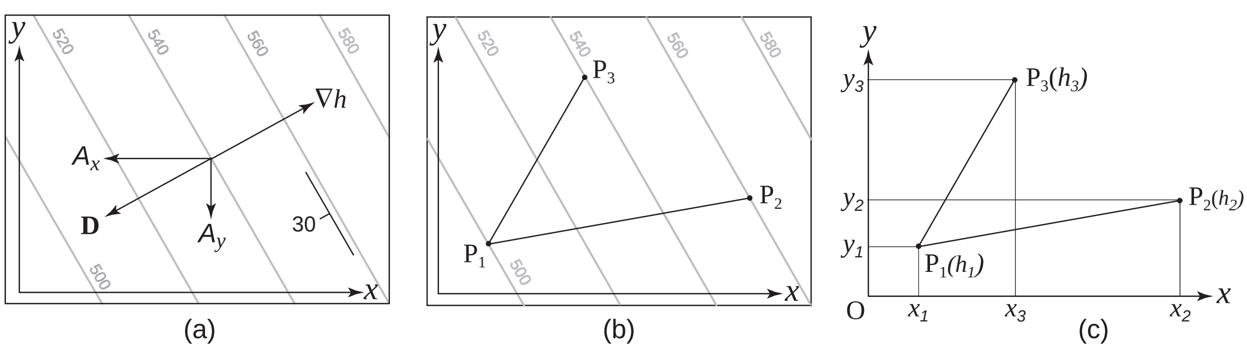

Figure 7.25: Structure contours and the gradient vector $\boldsymbol{\nabla} h$.

Because vector $\boldsymbol{\nabla} h$ is the steepest direction its magnitude is also the slope of the line of true dip. The dip direction is given by $-\boldsymbol{\nabla} h$, that is, opposite the direction of $\boldsymbol{\nabla} h$. The reason for the

[13]The vector operator symbol $\boldsymbol{\nabla}$ was introduced by the Irish mathematician and physicist William Rowen Hamilton [1805–1865], and called *nabla* after a Hebrew harp of similar shape. It is now commonly termed *del*, but do not confuse the name or symbol with the Greek *delta*.

[14]In this application the scalar field $h(x, y)$ describes an inclined plane and therefore the vector $\boldsymbol{\nabla} h$ has constant magnitude and direction everywhere in the field. But the analysis also applies to more general situations where $h(x, y)$ describes an curviplanar surface; $\boldsymbol{\nabla} h$ still exists at every point, but both its magnitude and direction will vary.

change of sign is that $\boldsymbol{\nabla} h$ refers to the maximum *increase* while the dip refers to the maximum *decrease* of h. We can avoid this minus sign by defining the dip vector as

$$\mathbf{D} = -\boldsymbol{\nabla} h. \tag{7.66}$$

The components of this vector in each of the coordinate directions give the magnitudes of the apparent dip vectors $\mathbf{A}_x$ and $\mathbf{A}_y$ (Fig. 7.245). Thus

$$A_x = -\partial h/\partial x \quad \text{and} \quad A_y = -\partial h/\partial y. \tag{7.67}$$

The magnitude of the dip vector is then

$$D = \sqrt{A_x^2 + A_y^2}, \tag{7.68}$$

and the angle of true dip is

$$\delta = \arctan D. \tag{7.69}$$

The angle vector $\mathbf{D}$ makes with $+x$ is given by

$$\theta_x = \arctan(A_y/A_x). \tag{7.70}$$

In order to find the gradient vector $\boldsymbol{\nabla} h$ we need to express its components in terms of the coordinates of the three known points $P_1(x_1, y_1, h_1)$, $P_2(x_2, y_2, h_2)$ and $P_3(x_3, y_3, h_3)$ on the plane. We may relate these components to the horizontal and vertical distances between pairs of these known points, and we do this for lines P_1P_2 and P_1P_3 (Fig. 7.25b).

1. The vertical distance between points P_1 and P_2 is $\Delta h_{12} = (h_2 - h_1)$. This is made up of two parts: Δh_x is associated with line parallel to the x axis and Δh_y is associated with line parallel to the y axis (Fig. 7.25c). In terms of the coordinates of points P_1 and P_2 these are

$$\Delta h_x = \frac{\partial h}{\partial x}(x_2 - x_1) \quad \text{and} \quad \Delta h_y = \frac{\partial h}{\partial y}(y_2 - y_1).$$

The total Δh is the sum of these two parts

$$\Delta h_{12} = \frac{\partial h}{\partial x}(x_2 - x_1) + \frac{\partial h}{\partial y}(y_1 - y_2) = (h_2 - h_1). \tag{7.71a}$$

2. Similarly, the vertical distance between points P_1 and P_3 is $\Delta h_{13} = (h_3 - h_1)$. It too is made up of two parts and the sum of these parts is

$$\Delta h_{13} = \frac{\partial h}{\partial x}(x_3 - x_1) + \frac{\partial h}{\partial y}(y_3 - y_1) = (h_3 - h_1). \tag{7.71b}$$

We now have two equations for the two unknown slopes $\partial h/\partial x$ and $\partial h/\partial y$. Solving for these using Cramer's rule[15] gives

$$\frac{\partial h}{\partial x} = \frac{\begin{vmatrix} (h_2 - h_1) & (y_2 - y_1) \\ (h_3 - h_1) & (y_3 - y_1) \end{vmatrix}}{\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix}} \quad \text{and} \quad \frac{\partial h}{\partial y} = \frac{\begin{vmatrix} (x_2 - x_1) & (h_2 - x_1) \\ (x_3 - x_1) & (h_3 - h_1) \end{vmatrix}}{\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix}}.$$

[15] Named after the Swiss mathematician Gabriel Cramer [1704–1752], a contemporary of Leonard Euler.

	x	y	h
P_1	100 m	60 m	535 m
P_2	350 m	16 m	415 m
P_3	156 m	214 m	440 m

Table 7.5: Dip vector from $\boldsymbol{\nabla} h$.

Expanding these determinants we have

$$\frac{\partial h}{\partial x} = \frac{(h_2 - h_1)(y_3 - y_1) - (h_3 - h_1)(y_2 - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}, \tag{7.72a}$$

$$\frac{\partial h}{\partial y} = \frac{(h_3 - h_1)(x_2 - x_1) - (h_2 - h_1)(x_3 - x_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}. \tag{7.72b}$$

Problem

- Solve the same three-point problem using this vector approach (see Fig. 7.25 and Table 7.5).

Solution

1. With coordinates (x, y) and heights h, Eqs. 7.66 yield the downward slopes in each coordinate direction $A_x(0.55317)$ and $A_y(0.41573)$. The corresponding dip angles are $\alpha_x = 28.95°$ and $\alpha_y = 22.57°$.

2. From Eqs. 7.67 and 7.68, $D = 0.69197$, hence the dip of the plane $\delta = 34.68°$.

3. From Eq. 7.69, $\theta_x = 36.93°$ and this is the trend of the direction of true dip measured from east.

Answer

- The attitude of the dip vector is $\mathbf{D}(35/053)$, which is the same as obtained by coordinate geometry (see Fig. 7.24c).

7.11 EXERCISES

1. Determine the direction angles of $\mathbf{L}(45/100)$ graphically on the stereonet. Compute the corresponding direction cosines and check the accuracy of your measurements with Eq. 7.5. Also compute the direction cosines from the plunge and trend and compare with your results.

2. Using direction angles plot vector $\mathbf{V}(-0.43301, -0.25000, 0.86603)$. Read the plunge and trend and compare with the results obtained with Eqs. 7.8.

3. The poles of two plane are $\mathbf{P}_1(30/050)$ and $\mathbf{P}_2(40/345)$. Graphically and analytically determine the dihedral angle between these planes and the attitude of the line of intersection.

4. Rotate the vector of Question No. 1 $\omega = 60^\circ$ about the x axis.

5. Given the two planes N 30 W, 20 E and N 50 E, 30 N determine the attitude of the line of intersection, the dihedral angle between the two planes and orientation of the bisector of the two planes.

6. MORE

7. MORE

8. MORE

9. MORE

10. MORE.